

POSITIVE FRAGMENTS OF COALGEBRAIC LOGICS

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ABSTRACT. Positive modal logic was introduced in an influential 1995 paper of Dunn as the positive fragment of standard modal logic. His completeness result consists of an axiomatization that derives all modal formulas that are valid on all Kripke frames and are built only from atomic propositions, conjunction, disjunction, box and diamond.

In this paper, we provide a coalgebraic analysis of this theorem, which not only gives a conceptual proof based on duality theory, but also generalizes Dunn's result from Kripke frames to coalgebras of weak-pullback preserving functors.

Among the category theoretic results we prove to facilitate this analysis are the following. Every functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ has a \mathbf{Pos} -enriched left Kan extension $\mathbf{Pos} \rightarrow \mathbf{Pos}$. A functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ is a \mathbf{Pos} -enriched left Kan extension of a functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ if and only if it preserves 'truncated nerves of posets'. An endofunctor on an ordered variety has a presentation by monotone operations and equations if and only if it preserves \mathbf{Pos} -enriched sifted colimits.

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1. INTRODUCTION

Consider modal logic as given by atomic propositions, Boolean operations, and a unary box, together with its usual axiomatisation stating that box preserves finite meets. In [18], Dunn answered the question of an axiomatisation of the positive fragment of this logic, where the positive fragment is given by atomic propositions, lattice operations, and unary box and diamond (but no negation).

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Here we seek to generalize this result from Kripke frames to coalgebras for a weak pullback preserving functor. Whereas Dunn had no need to justify that the positive fragment actually *adds* a modal operator (the diamond), the general situation requires a conceptual clarification of this step. And, as it turns out, what looks innocent enough in the familiar case is at the heart of the general construction.

In the general case, we start with a functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$. From T we can obtain by duality a functor $L : \mathbf{BA} \rightarrow \mathbf{BA}$ on the category \mathbf{BA} of Boolean algebras, so that the free L -algebras are exactly the Lindenbaum algebras of the modal logic. We are going to take the functor L itself as the category theoretic counterpart of the corresponding modal logic. How should we construct the positive T -logic? Dunn gives us a hint in that he notes that in the same way as standard modal logic is given by algebras over \mathbf{BA} , positive modal logic is given by algebras over the category \mathbf{DL} of (bounded) distributive lattices. It follows that the positive fragment of (the logic corresponding to) L should be a functor $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ which, in turn, by duality, should arise from a functor $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ on the category \mathbf{Pos} of posets and monotone maps.

The centrepiece of our construction is now the observation that any functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ has a canonical extension to a functor $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$. Theorem 6.12 then shows that this construction $T \mapsto T' \mapsto L'$ indeed gives the positive fragment of L and so generalizes Dunn's theorem.

An important observation about the positive fragment is the following: given any Boolean formula, we can rewrite it as a positive formula with negation only appearing on atomic propositions. In other words, the translation β from positive logic to Boolean logic given by

$$\begin{aligned} (1) \quad & \beta(\Diamond\phi) = \neg\Box\neg\beta(\phi) \\ (2) \quad & \beta(\Box\phi) = \Box\beta(\phi) \end{aligned}$$

induces a bijection (on equivalence classes of formulas taken up to logical equivalence). More algebraically, we can formulate this as follows.

Given a Boolean algebra $B \in \mathbf{BA}$, let LB be the free Boolean algebra generated by $\{\Box b \mid b \in B\}$ modulo the axioms of modal logic. Given a distributive lattice A , let $L'A$ be the free distributive lattice generated by $\{\Box a : a \in A\} \cup \{\Diamond a \mid a \in A\}$ modulo the axioms of positive modal logic. Further, let us denote by $W : \mathbf{BA} \rightarrow \mathbf{DL}$ the forgetful functor. Then the above observation that every modal formula can be written, up to logical equivalence, as a positive modal formula with negations pushed to atoms, can be condensed into the statement that the (natural) distributive lattice homomorphism

$$(3) \quad \beta_B : L'WB \rightarrow WLB$$

induced by (1), (2) is an isomorphism.

Our main results, presented in Sections 6 and 7, are the following. If T' is an extension of T and L, L' are the induced logics, then $\beta : L'W \rightarrow WL$ exists. If, moreover, T' is the induced extension (posetification) of T and T preserves weak pullbacks, then β is an isomorphism (Theorem 6.12). Furthermore, in the same way as the induced logic L can be seen as the logic of all predicate liftings of T , the induced logic L' is the logic of all *monotone* predicate liftings of T (Theorem 7.2).

These results depend crucially on the fact that the posetification T' of T is defined as a completion with respect to \mathbf{Pos} -enriched colimits. We devote Section 4 to establishing some results on posetifications used later. To show that the posetification always exists, we prove that any functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ extends canonically to a locally monotone functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ (Theorem 4.3). Moreover, we characterize those functors $\mathbf{Pos} \rightarrow \mathbf{Pos}$ that arise as such extensions as the functors that preserve ‘truncated nerves of posets’ (Theorem 4.13).

On the algebraic side the move to \mathbf{Pos} -enriched colimits guarantees that the modal operations are monotone. In Section 5, and recalling [29, Theorem 4.7] stating that a functor $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ preserves ordinary sifted colimits if and only if it has a presentation by operations and equations, we show here that $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ preserves *enriched* sifted colimits if and only if it has a presentation by *monotone* operations and equations (Theorem 5.16). To see the relevance of a presentation result specific to monotone operations, observe that in the example of positive modal logic it is indeed the case that both \Box and \Diamond are monotone.

2. ON COALGEBRAS AND COALGEBRAIC LOGIC

A Kripke model (W, R, v) (see eg [10] for an introduction to modal logic) with $R \subseteq W \times W$ and $v : W \rightarrow 2^{\mathbf{AtProp}}$ can also be described as a coalgebra $W \rightarrow \mathcal{P}W \times 2^{\mathbf{AtProp}}$, where $\mathcal{P}W$ stands for the powerset of W . This point of view suggests to generalize modal logic from Kripke frames to coalgebras

$$\xi : X \rightarrow TX$$

where T may now be any functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$. We recover Kripke models by putting $TX = \mathcal{P}X \times 2^{\mathbf{AtProp}}$. We also recover the so-called bounded morphisms or p-morphisms as coalgebras morphisms $f : (X, \xi) \rightarrow (X', \xi')$, that is, as maps $f : X \rightarrow X'$ such that $Tf \circ \xi = \xi' \circ f$.

2.A. Coalgebras and algebras. More generally, for any category \mathcal{C} and functor $T : \mathcal{C} \rightarrow \mathcal{C}$, we have the category $\mathbf{Coalg}(T)$ of T -coalgebras with objects and morphisms as above. Dually, $\mathbf{Alg}(T)$ is the category where the objects $\alpha : TX \rightarrow X$ are arrows in \mathcal{C} and where the morphisms $f : (X, \alpha) \rightarrow (X', \alpha')$ are arrows $f : X \rightarrow X'$ in \mathcal{C} such that $f \circ \alpha = \alpha' \circ Tf$. It is worth noting that T -coalgebras over \mathcal{C} are dual to $T^{\mathbf{op}}$ -algebras over $\mathcal{C}^{\mathbf{op}}$, that is, $\mathbf{Coalg}(T)^{\mathbf{op}} = \mathbf{Alg}(T^{\mathbf{op}})$. Here $\mathcal{C}^{\mathbf{op}}$ is the category which has the same objects and arrows as \mathcal{C} but domain and codomain of arrows interchanged and $T^{\mathbf{op}} : \mathcal{C}^{\mathbf{op}} \rightarrow \mathcal{C}^{\mathbf{op}}$ is the functor that has the same action on objects and morphisms as T .

2.B. Duality of Boolean algebras and sets. The abstract duality between algebras and coalgebras becomes particularly interesting if we put it on top of a concrete duality, such as the dual adjunction between the category \mathbf{Set} of sets and functions and the category \mathbf{BA} of Boolean algebras. We denote by $P : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{BA}$ the functor taking powersets and by $S : \mathbf{BA} \rightarrow \mathbf{Set}^{\mathbf{op}}$ the functor taking ultrafilters. Alternatively, we can describe these functors by $PX = \mathbf{Set}(X, 2)$ and $SA = \mathbf{BA}(A, 2)$, which also determines their action on arrows (here 2 denotes the two-element Boolean algebra). P and S are adjoint, satisfying $\mathbf{Set}(X, SA) \cong \mathbf{BA}(A, PX)$. Restricting P and S to finite Boolean algebras/sets, this adjunction becomes a dual equivalence [21, VI.(2.3)].

2.C. Boolean logics for coalgebras, syntax. What now are logics for coalgebras? We follow a well-established methodology in modal logic [10] and study modal logics via the associated category of modal algebras. More formally, given a modal logic \mathcal{L} extending Boolean propositional logic and with associated category \mathcal{A} of modal algebras, we describe \mathcal{L} by a functor

$$L : \mathbf{BA} \rightarrow \mathbf{BA}$$

so that the category $\mathbf{Alg}(L)$ of algebras for the functor L coincides with \mathcal{A} . In particular, the Lindenbaum algebra of \mathcal{L} will be the initial L -algebra.

Example 2.1. Let T be the powerset functor and $L : \mathbf{BA} \rightarrow \mathbf{BA}$ be the functor mapping an algebra A to the algebra LA generated by $\Box a$, $a \in A$, and quotiented by the relation stipulating that \Box preserves finite meets, that is,

$$(4) \quad \Box \top = \top \quad \Box(a \wedge b) = \Box a \wedge \Box b$$

$\mathbf{Alg}(L)$ is the category of modal algebras (Boolean algebras with operators), a result which appears to be explicitly stated first in [1].

2.D. Boolean logics for coalgebras, semantics. The semantics of such a logic is described by a natural transformation

$$\delta : LP \rightarrow PT^{\text{op}}$$

Intuitively, each modal operator in LPX is assigned its meaning as a subset of TX . More formally, δ allows us to lift $P : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{BA}$ to a functor

$$\begin{aligned} \mathbf{Coalg}(T) &\xrightarrow{P^\sharp} \mathbf{Alg}(L) \\ (X, \xi) &\mapsto P^\sharp(X, \xi) = (PX, LPX \xrightarrow{\delta_X} PTX \xrightarrow{P\xi} PX) \end{aligned}$$

If we consider a formula ϕ to be an element of the initial L -algebra (the Lindenbaum algebra of the logic), then the semantics of ϕ as a subset of a coalgebra (X, ξ) is given by the unique arrow from that initial algebra to $P^\sharp(X, \xi)$.

Remark 2.2. This account of the semantics of modal logic is typical for the coalgebraic approach. One first defines a one-step semantics relating formulas with precisely one layer of modal operators (as described by L) with one step of transitions on the semantic side (as described by T). Then one uses coinduction, or, as in this case, induction in order to extend the ‘one-step situation’ to arbitrary formulas and behaviors.

Example 2.3. For the powerset functor we define the (one-step) semantics $\delta_X : LPX \rightarrow PP^{\text{op}}X$ by

$$(5) \quad \Box a \mapsto \{b \subseteq X \mid b \subseteq a\}, \text{ for } a \in PX.$$

It is an old result in domain theory that δ_X is an isomorphism for finite X , see [1]. This implies completeness of the axioms (4) with respect to Kripke semantics (5).

2.E. Functors having presentations by operations and equations. One might ask when a functor $L : \mathbf{BA} \rightarrow \mathbf{BA}$ can legitimately be considered to give rise to a modal logic. For us, in this paper, a minimal requirement on L is that $\mathbf{Alg}(L)$ is a variety in the sense of universal algebra, that is, that $\mathbf{Alg}(L)$ can be described by operations and equations, the operations then corresponding to modal operators and the equations to axioms. This happens if L is determined by its action on

finitely generated free algebras, see [29]. These functors are also characterized as functors having presentations by operations and equations, or as functors preserving sifted colimits. Most succinctly, they are precisely those endofunctors on \mathbf{BA} that arise as left Kan extensions of their restrictions along the inclusion functor $\mathbf{BA}_{\text{ff}} \rightarrow \mathbf{BA}$ of the full subcategory \mathbf{BA}_{ff} of \mathbf{BA} consisting of free algebras on finitely many generators.

2.F. The (finitary, Boolean) coalgebraic logic of a Set-functor. The general considerations laid out above suggest to define the finitary (Boolean) coalgebraic logic associated to a given functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ as

$$(6) \quad \mathbf{L}Fn = PT^{\text{op}}SF n$$

where Fn denotes the free Boolean algebra over n generators, for n ranging over natural numbers. The semantics δ is given by observing that natural transformations $\delta : \mathbf{L}P \rightarrow PT^{\text{op}}$ are in bijection with natural transformations

$$(7) \quad \hat{\delta} : \mathbf{L} \rightarrow PT^{\text{op}}S$$

and we can let $\hat{\delta}$ to be the identity on finitely generated free algebras.

More explicitly, $\mathbf{L}A$ can be represented as the free Boolean algebra over

$$\{\sigma(a_1, \dots, a_n) \mid \sigma \in PT^{\text{op}}SF n, a_i \in A, n < \omega\}$$

modulo appropriate axioms, with $\delta_X : \mathbf{L}PX \rightarrow PT^{\text{op}}X$ given by $\delta\sigma(a_1, \dots, a_n) = PT^{\text{op}}(\hat{a})(\sigma)$ where $\hat{a} : X \rightarrow SF n$ is the adjoint transpose of $(a_1, \dots, a_n) : n \rightarrow UPX$, with the forgetful functor $U : \mathbf{BA} \rightarrow \mathbf{Set}$ being right adjoint of F .¹ Of course, in concrete examples one is often able to obtain much more succinct presentations:

Proposition 2.4. *For T the powerset functor, the functor \mathbf{L} defined by (6) is isomorphic to the functor L of Example 2.1.*

Proof. In analogy with (7), let $\hat{\delta} : L \rightarrow PT^{\text{op}}S$ be the transpose of $\delta : LP \rightarrow PT^{\text{op}}$ as defined in (5). We know from Example 2.3 that $\hat{\delta}_{Fn} : LFn \rightarrow PT^{\text{op}}SF n = \mathbf{L}Fn$ is an isomorphism. But as both L and \mathbf{L} are determined by their action on finitely generated free algebras, this extends to an isomorphism $L \rightarrow \mathbf{L}$. \square

Remark 2.5. \mathbf{L} is universal in the sense that any other finitary Boolean coalgebraic logic L for T is uniquely determined by the natural transformation $L \rightarrow \mathbf{L}$ constructed in the proof above. More formally, we can express this universality as follows: denote by $\text{Sift}[\mathbf{BA}, \mathbf{BA}]$ the category of sifted-colimit-preserving functors from \mathbf{BA} to \mathbf{BA} . Then $\hat{\delta} : \mathbf{L} \rightarrow PT^{\text{op}}S$ as given in (7) is final in the slice category $\text{Sift}[\mathbf{BA}, \mathbf{BA}]/PT^{\text{op}}S$.

Proposition 2.4 can be understood as saying that the logic defined by finality as above has a simple concrete presentation given by (4) and (5).

¹Since elements in $PT^{\text{op}}SF n$ are in one-to-one correspondence with natural transformations $\text{Set}(-, 2^n) \rightarrow \text{Set}(T-, 2)$, also known as predicate liftings [36], we see that the logic L coincides with the logic of all predicate liftings of [38], with the difference that L also incorporates axioms. The axioms are important to us as otherwise the natural transformation β , see (3), mentioned in the introduction might not exist.

2.G. Outlook: Positive coalgebraic logic. It is evident that, at least for some of the developments above, not only the functor T , but also the categories \mathbf{Set} and \mathbf{BA} can be considered to be parameters. Accordingly, one expects that positive coalgebraic logic takes place over the category \mathbf{DL} of (bounded) distributive lattices which in turn, is part of an adjunction $P' : \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{DL}$, taking upsets, and $S' : \mathbf{DL} \rightarrow \mathbf{Pos}^{\text{op}}$, taking prime filters, or, equivalently, $P'X = \mathbf{Pos}(X, 2)$ and $S'A = \mathbf{DL}(A, 2)$ where 2 is, as before, the two-element chain (now considered, depending on the context, either as a poset or as a distributive lattice). Consequently, the ‘natural semantics’ of positive logics is ‘ordered Kripke frames’, or coalgebras over posets.

Replaying the developments above with \mathbf{Pos} and \mathbf{DL} instead of \mathbf{Set} and \mathbf{BA} , we may define a logic for T' -coalgebras, with $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$, to be given by a natural transformation

$$(8) \quad \delta' : \mathbf{L}'P' \rightarrow P'T'^{\text{op}}$$

where

$$(9) \quad \mathbf{L}'F'Dn = P'T'^{\text{op}}S'F'Dn$$

is a functor determined *by its action on finitely discretely generated free distributive lattices* and δ' is given by its transpose in the same way as in (7). Here $D : \mathbf{Set} \rightarrow \mathbf{Pos}$ denotes the functor equipping a set with the discrete order.

Example 2.6. Given a poset X , a subset $Y \subseteq X$ is called *convex* if $y \leq y' \leq y''$ and $y, y'' \in Y$ imply $y' \in Y$. The convex powerset functor $\mathcal{P}' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ maps a poset to the set of its convex subsets, ordered by the Egli-Milner order, and a monotone map to its direct image. Let now $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ be the functor mapping a distributive lattice A to the distributive lattice $L'A$ generated by $\Box a$ and $\Diamond a$ for all $a \in A$, and subject to the relations stipulating that \Box preserves finite meets, \Diamond preserves finite joins, and

$$(10) \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \quad \Box(a \vee b) \leq \Diamond a \vee \Box b$$

The natural transformation $\delta'_X : \mathbf{L}'P'X \rightarrow P'\mathcal{P}'^{\text{op}}X$ is defined by

$$(11) \quad \Diamond a \mapsto \{b \subseteq X \mid b \text{ is a convex subset in } X \text{ and } b \cap a \neq \emptyset\}, \text{ for } a \in P'X,$$

the clause for $\Box a$ being the same as in (5).

Remark 2.7. $\mathbf{Alg}(L')$ is the category of positive modal algebras of Dunn [18]. We shall later see in Corollary 5.17 that it is isomorphic to $\mathbf{Alg}(\mathbf{L}')$. We have again that for finite X , δ'_X is an isomorphism, a representation first stated in [21, 22], the connection with modal logic being given by [1, 37, 42] and investigated from a coalgebraic point of view in [35]. As opposed to [35], we take the set-theoretic semantics of modal logic as fundamental and do not have to use the topological semantics based on Stone or Priestley duality: all we need is contained in the adjunctions $S \dashv P : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{BA}$ and $S' \dashv P' : \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{DL}$.

2.H. Outlook: Coalgebraic logic enriched over \mathbf{Pos} . Moving from ordinary categories to categories enriched over \mathbf{Pos} plays a major role in this paper. From the point of view of our application, positive modal logic, the reason is that enrichment over \mathbf{Pos} takes automatically care of the fact that positive modal logics extend the logic of distributive lattices by *monotone* modal operations. Throughout the paper, we shall encounter many more reasons on the technical level, some of which are the following.

- (1) The category \mathbf{Pos} is the cocompletion by *enriched* sifted colimits of the category of finite sets.
- (2) The posetification is doing the what we want it to do only if it is defined in terms of *enriched* left Kan extensions.
- (3) Among all functors on \mathbf{Pos} , posetifications are characterized by ‘coinserters of truncated nerves’ where a coinserter is the enriched analogue of coequalizers.
- (4) In the ordered setting, one is frequently interested in definability by inequations (\leq) instead of definability by equations and quotienting by inequations corresponds to taking a coinserter, not to taking a coequalizer.
- (5) Having a presentation by *monotone* operations and equations in *discrete arities* is characterized by preservation of *enriched* sifted colimits.

3. ON \mathbf{Pos} AND \mathbf{Pos} -ENRICHED CATEGORIES

Below we recall some notions of enriched category theory needed in the sequel. Most of this section is rather technical, but we have decided to include it in order to keep the paper self-contained. However, for more details, we refer the reader to Kelly’s monograph [23].

3.A. The category \mathbf{Pos} of posets and monotone maps. The category \mathbf{Pos} has partial orders (posets) as objects and monotone maps as arrows. \mathbf{Pos} is complete and cocomplete (even locally finitely presentable [2]). Limits are computed as in \mathbf{Set} , with the order on the limit being the largest relation making the maps in the cocone monotone. Colimits are easiest to compute in two steps. First, colimits in the category of preorders are computed as in \mathbf{Set} , with the preorder on the colimit being the smallest one making the maps in the cocone monotone. Second, one quotients the preorder by anti-symmetry in order to obtain a poset (directed colimits, however, are computed as in \mathbf{Set} , see [2]). \mathbf{Pos} is also cartesian closed, with the internal hom $[X, Y]$ being the poset of monotone maps from X to Y , ordered pointwise.

3.B. \mathbf{Pos} -enriched categories. We shall consider categories *enriched* in \mathbf{Pos} . Thus, a \mathbf{Pos} -enriched category \mathcal{C} is a category with ordered homsets, such that composition is monotone in both arguments: $g \circ f \leq k \circ h$ whenever $g \leq k$ and $f \leq h$; a \mathbf{Pos} -enriched functor $T : \mathcal{C} \rightarrow \mathcal{D}$ is a locally monotone functor, that is, it preserves the order on the homsets: $f \leq g$ implies $Tf \leq Tg$. A \mathbf{Pos} -natural transformation between locally monotone functors is just a natural transformation, the extra condition of enriched naturality being vacuous here. The category of \mathbf{Pos} -enriched functors from \mathcal{C} to \mathcal{D} and natural transformations between them will be denoted by $[\mathcal{C}, \mathcal{D}]$. The opposite category \mathcal{C}^{op} of \mathcal{C} has just the sense of morphisms reversed, the order on hom-posets remains unchanged.

Besides \mathbf{Pos} itself, an example of a \mathbf{Pos} -enriched category is \mathbf{Set} , the category of sets and functions, considered discretely enriched. In the sequence of adjunctions $\mathcal{C} \dashv D \dashv V : \mathbf{Pos} \rightarrow \mathbf{Set}$ between the connected components functor, the discrete functor and the forgetful one, only the adjunction $\mathcal{C} \dashv D : \mathbf{Set} \rightarrow \mathbf{Pos}$ is enriched; in particular the discrete functor $D : \mathbf{Set} \rightarrow \mathbf{Pos}$ is locally monotone, while the forgetful functor $V : \mathbf{Pos} \rightarrow \mathbf{Set}$ fails to be so. Also, due to the discrete enrichment, any functor $\mathbf{Set} \rightarrow \mathbf{Set}$ is locally monotone.

3.C. Weighted (co)limits; coinserter; Kan-extensions. Recall from [23] that the proper concepts of limits and colimits in enriched category theory are those of weighted (co)limits. Specifically, the colimit of a \mathbf{Pos} functor $H : \mathcal{K} \rightarrow \mathcal{C}$ weighted by a \mathbf{Pos} -functor $W : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Pos}$ is an object $W * H$ in \mathcal{C} , together with an isomorphism

$$\mathcal{C}(W * H, X) \cong [\mathcal{K}^{\text{op}}, \mathbf{Pos}](W, \mathcal{C}(H-, X))$$

of posets, natural in $X \in \mathcal{C}$. Dually, a limit of $H : \mathcal{K} \rightarrow \mathcal{C}$ weighted by $W : \mathcal{K} \rightarrow \mathbf{Pos}$ is an object $\{W, H\}$ in \mathcal{C} , together with an isomorphism

$$\mathcal{C}(X, \{W, H\}) \cong [\mathcal{K}, \mathbf{Pos}](\mathcal{C}(X, H-), W)$$

of posets, again natural in $X \in \mathcal{C}$.

One important example of weighted (co)limits are (co)powers, which arise from constant weights and diagrams. Specifically, the copower $X \bullet C$ of a poset $X \in \mathbf{Pos}$ and an object $C \in \mathcal{C}$ is characterized by the isomorphism $\mathcal{C}(X \bullet C, -) \cong [X, \mathcal{C}(C, -)]$, while the power of $X \in \mathbf{Pos}$ and $C \in \mathcal{C}$, denoted $X \pitchfork C$, satisfies $\mathcal{C}(-, X \pitchfork C) \cong [X, \mathcal{C}(-, C)]$ [23, Section 3.7].

Another example of weighted (co)limit that will later appear in the paper is the (co)inserter:

Example 3.1. [25] A *coinserter* is a colimit that has weight $W : \mathcal{K}^{\text{op}} \rightarrow \mathbf{Pos}$, where \mathcal{K} is

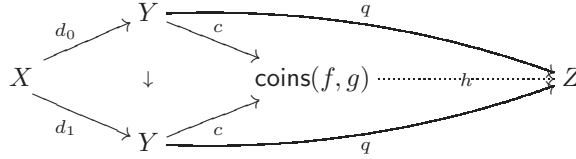
$$\cdot \rightrightarrows \cdot$$

and is mapped by W to the parallel pair

$$2 \xrightleftharpoons[0]{1} 1$$

in \mathbf{Pos} , with arrow 0 mapping to $0 \in 2$ and arrow 1 mapping to $1 \in 2$ (recall that 2 is the poset $\{0 \leq 1\}$). A functor F from \mathcal{K} to a \mathbf{Pos} -category \mathcal{C} corresponds to a parallel pair of arrows $d_0, d_1 : X \rightrightarrows Y$ in \mathcal{C} .

In detail, the coinserter of d_0, d_1 consists of an object $\text{coins}(d_0, d_1)$, and an arrow $c : Y \rightarrow \text{coins}(d_0, d_1)$ with $c \circ d_0 \leq c \circ d_1$, having the following universal property: for any $q : Y \rightarrow Z$ with $q \circ d_0 \leq q \circ d_1$, there is a unique $h : \text{coins}(f, g) \rightarrow Z$ with $h \circ c = q$. Moreover, this assignment is monotone, in the sense that given $q, q' : Y \rightarrow Z$ with $q \leq q'$, $q \circ d_0 \leq q \circ d_1$ and $q' \circ d_0 \leq q' \circ d_1$, the corresponding unique arrows $h, h' : \text{coins}(f, g) \rightarrow Z$ satisfy $h \leq h'$.



The coinserter is called *reflexive* if d_0 and d_1 have a common right inverse $i : Y \rightarrow X$; that is, $d_0 \circ i = d_1 \circ i = \text{id}_Y$.

By reversing the direction of the arrows, one obtains the dual notion of a (coreflexive) inserter. \square

Remark 3.2. Informally speaking, coinserter take quotients with respect to preorders, whereas coequalizers take quotients with respect to equivalence relations. For later use, we recall how coinserter are built in \mathbf{Pos} . For a pair of monotone maps $d_0, d_1 : X \rightarrow Y$, define first a binary relation \mathbf{r} on the underlying set of the

poset Y as follows: given $y, y' \in Y$, say that $y \mathbf{r} y'$ if there are $x_0, \dots, x_n \in X$ such that

$$\begin{array}{ccccccc} & x_0 & & x_1 & & \dots & & x_n \\ & \swarrow d_0 \quad \searrow d_1 & & \swarrow d_0 \quad \searrow d_1 & & & & \swarrow d_0 \quad \searrow d_1 \\ y \leq d_0(x_0) & & d_1(x_0) \leq d_0(x_1) & & d_1(x_1) \leq \dots \leq d_0(x_n) & & & d_1(x_n) \leq y' \end{array}$$

It is easy to see that \mathbf{r} is a reflexive and transitive relation, thus a preorder on Y . Then the coinserter of d_0 and d_1 is the quotient of Y with respect to the equivalence relation induced by \mathbf{r} , with order $[y] \leq [y']$ iff $y \mathbf{r} y'$. \square

The importance of reflexive coinserter for us stems from the fact that each poset can be canonically expressed as a reflexive coinserter of *discrete* posets:

Proposition 3.3. *Let X be a poset. Denote by X_0 its underlying set, and by X_1 the set of all comparable pairs, $X_1 = \{(x, x') \in X \mid x \leq x'\}$. Let $d_0, d_1 : X_1 \rightarrow X_0$ be the maps $d_0(x, x') = x$, $d_1(x, x') = x'$, with common right inverse $i : X_0 \rightarrow X_1$, $i(x) = (x, x)$. Then the obvious (monotone) map $c : DX_0 \rightarrow X$, $c(x) = x$, exhibits X as the coinserter in \mathbf{Pos} of the reflexive pair of discrete posets (also called the truncated nerve of the poset)*

$$(12) \quad N_X : DX_1 \begin{array}{c} \xrightarrow{Dd_0} \\ \xleftarrow{Dd_1} \end{array} DX_0 \xrightarrow{c} X$$

$\underbrace{\hspace{10em}}_{Di}$

Proof. We leave to the reader to check the straightforward details. \square

Definition 3.4. Let $J : \mathcal{K} \rightarrow \mathcal{C}$, $H : \mathcal{K} \rightarrow \mathcal{D}$ be locally monotone functors. A \mathbf{Pos} -enriched *left Kan extension of H along J* , is a locally monotone functor $\text{Lan}_J H : \mathcal{C} \rightarrow \mathcal{D}$, such that there is a \mathbf{Pos} -natural isomorphism

$$\text{Lan}_J H(C) \cong \mathcal{C}(J-, C) * H$$

for each C in \mathcal{C} .

Remark 3.5.

- (1) For any locally monotone functor $\tilde{H} : \mathcal{C} \rightarrow \mathcal{D}$, there is an isomorphism

$$(13) \quad [\mathcal{C}, \mathcal{D}](\text{Lan}_J H, \tilde{H}) \cong [\mathcal{K}, \mathcal{D}](H, \tilde{H}J)$$

in analogy to the case of unenriched left Kan extensions. In particular, there is a \mathbf{Pos} -natural transformation $(\text{Lan}_J H)J \rightarrow H$ which is universal in the sense that for a locally monotone functor $\tilde{H} : \mathcal{C} \rightarrow \mathcal{D}$, any \mathbf{Pos} -natural transformation $H \rightarrow \tilde{H}J$ factorizes through $\text{Lan}_J H$. In the general enriched setting, requiring this isomorphism is *strictly weaker* than the above definition, but if \mathcal{D} is *powered*, it can however be taken as an alternative definition of left Kan extensions (see the discussion after Equation (4.45) in [23]).

- (2) Suppose $J : \mathcal{K} \rightarrow \mathcal{C}$ is fully faithful. Then the \mathbf{Pos} -natural transformation $H \rightarrow (\text{Lan}_J H)J$ induced by the isomorphism (13) is an isomorphism [23, Proposition 4.23].
- (3) By general enriched category theory, the \mathbf{Pos} -enriched left Kan extension $\text{Lan}_J H$ exists whenever \mathcal{K} is small and \mathcal{D} is cocomplete. But it might exist even when \mathcal{K} is not small, as we shall see later in a special case (Thm. 4.3).

- (4) From (13) it follows that any locally monotone left adjoint $Q : \mathcal{D} \rightarrow \mathcal{E}$ preserves the \mathbf{Pos} -enriched left Kan extension $\text{Lan}_J H$, in the sense that $\text{Lan}_J(QH) \cong Q \text{Lan}_J H$.

Example 3.6. Let $D : \mathbf{Set} \rightarrow \mathbf{Pos}$ be the discrete functor and $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ the powerset functor. Then the \mathbf{Pos} -enriched left Kan extension of $D\mathcal{P}$ along D is the convex powerset functor [30]. On the other hand, the ordinary left Kan extension of $D\mathcal{P}$ along D is DPV ,² which is less interesting as it maps any poset to the discrete poset of its subsets. Remark 4.5 will give some more details.

3.D. Ordered varieties. We have seen in Section 2 a close interplay between modal logic and varieties of algebras. The theory of (locally monotone) \mathbf{Pos} -functors and their logics of monotone modal operators naturally leads to the world of ordered varieties, as defined by Bloom and Wright in [12].

More precisely, recall that a signature Σ associates to each natural number n a *set* of operation symbols Σ_n of arity n . A Σ -algebra consists of a *poset* A and for each $\sigma \in \Sigma_n$, a *monotone* operation $\sigma_A : A^n \rightarrow A$. An ordered variety is specified by a signature Σ and a set of inequations. Bloom [11] proved that varieties are precisely the HSP closed subclasses of algebras for a signature, provided that we understand closure under H as closure under surjective homomorphisms and closure under S as closure under embeddings (injective and order-reflecting homomorphisms).

The structure theory of ordered varieties is similar to the one for ordinary varieties. For more details we refer the reader to the original [12] and to the more recent paper [32].

Example 3.7. The category \mathbf{BA} of Boolean algebras is a variety over \mathbf{Pos} if we take Boolean algebras to be discretely ordered. The category \mathbf{DL} of distributive lattices is a variety over \mathbf{Pos} if we take algebras to be ordered in the lattice order: $a \leq b \Leftrightarrow a \wedge b = a$.

Notice that Boolean algebras can only be discretely ordered, because of the requirement that operations of ordered algebras should be monotone. In the case of Boolean algebras it is not hard to show that the discrete order is the only one that makes all operations (including negation) monotone (see Section 5.C below).

3.E. Sifted weights and sifted colimits; strongly finitary functors. There is a well-known result that a finitary \mathbf{Set} -endofunctor also preserves sifted colimits, or equivalently, filtered colimits and reflexive coequalizers [3, Corollary 6.30]. Below we sketch the corresponding \mathbf{Pos} -enriched theory (for more details, we refer to [14, 26, 32, 33]).

A weight $W : \mathcal{K}^{op} \rightarrow \mathbf{Pos}$ is called *sifted* if finite products commute with W -colimits in \mathbf{Pos} [26]. Equivalently, if the 2-functor $W * - : [\mathcal{K}, \mathbf{Pos}] \rightarrow \mathbf{Pos}$ preserves finite products. A *sifted colimit* is a colimit weighted by a sifted weight. Examples of sifted colimits are filtered colimits and reflexive coequalizers, but also reflexive inserters (see [14]).

There is a close interplay between (ordered) varieties and (enriched) sifted colimits, see also Section 5. For now, remember that in the non-enriched setting, a functor on a variety preserves ordinary sifted colimits iff it preserves filtered colimits and reflexive coequalizers [29]. In the \mathbf{Pos} -enriched setting, a locally monotone

²As D has non-enriched right adjoint V .

functor on an ordered variety preserves (enriched) sifted colimits iff it preserves filtered colimits and reflexive coinserter [32, Proposition 6.8].

Let \mathbf{Set}_f be the category of finite sets and maps, and ι the inclusion $\mathbf{Set}_f \xrightarrow{I} \mathbf{Set} \xrightarrow{D} \mathbf{Pos}$. In [32], following [14, Theorem 8.4], it was noticed that \mathbf{Pos} is the free cocompletion of \mathbf{Set}_f under enriched sifted colimits.³ Briefly, it means that every poset can be expressed as a canonical filtered colimit of finite posets, which in turn arise as reflexive coinserter of discrete finite (po)sets.

Definition 3.8. ([26]) A *strongly finitary* functor $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ is a locally monotone functor isomorphic to the \mathbf{Pos} -enriched left Kan extension along ι of its restriction, that is $T' \cong \mathbf{Lan}_\iota(T'\iota)$.

Thus strongly finitary functors are precisely the locally monotone functors on \mathbf{Pos} preserving sifted colimits.

Recall the examples BA and DL of \mathbf{Pos} -categories. They are connected by the monadic enriched adjunctions $F \dashv U : \mathbf{BA} \rightarrow \mathbf{Set}$, $F' \dashv U' : \mathbf{DL} \rightarrow \mathbf{Pos}$, where U and U' are the corresponding forgetful functors. Let $\mathbf{J} : \mathbf{BA}_{\text{ff}} \rightarrow \mathbf{BA}$ and $\mathbf{J}' : \mathbf{DL}_{\text{ff}} \rightarrow \mathbf{DL}$ denote the inclusion functors of the full subcategories spanned by the algebras which are *free on finite (discrete po)sets*.

Lemma 3.9. \mathbf{J} and \mathbf{J}' exhibit BA, respectively DL, as the free enriched cocompletions under sifted colimits of \mathbf{BA}_{ff} and \mathbf{DL}_{ff} . In particular, these functors are dense.⁴

Proof. We know that the functor $\mathbf{J} : \mathbf{BA}_{\text{ff}} \rightarrow \mathbf{BA}$ exhibits BA as a free cocompletion under sifted colimits (see [29]). Now the conclusion follows because of the discrete enrichment.

For distributive lattices, the result is an instance of [32, Theorem 6.10], since DL is a finitary variety of ordered algebras (thus, DL is isomorphic to the category of algebras for a strongly finitary monad on \mathbf{Pos}). \square

Corollary 3.10. A functor $L : \mathbf{BA} \rightarrow \mathbf{BA}$ has the form $\mathbf{Lan}_{\mathbf{J}}(L\mathbf{J})$ iff it preserves (ordinary) sifted colimits. A functor $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ has the form $\mathbf{Lan}_{\mathbf{J}'}(L'\mathbf{J}')$ iff it preserves sifted colimits.

4. PRESENTING FUNCTORS ON \mathbf{Pos} BY OPERATIONS AND EQUATIONS

For reasons explained in the introduction, we are interested in the posetification of functors $T : \mathbf{Set} \rightarrow \mathbf{Set}$. Technically, they can be described as enriched left Kan extensions of the functors $DT : \mathbf{Set} \rightarrow \mathbf{Pos}$. This suggests to also investigate the more general question of when a left Kan extension of a functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ exists. For general reasons, we know that such a left Kan extension exists if the functor is finitary, but that would exclude the example $T = \mathcal{P}$ from the introduction. Therefore, in Section 4.A, we show that *any* functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ has an enriched left

³ Let Φ be a class of weights and \mathcal{C} a \mathbf{Pos} -category. Following [4], let $\Phi\text{-Cocts}$ be the 2-category of Φ -cocomplete categories, Φ -cocontinuous functors, and natural transformations. The free cocompletion of \mathcal{C} with respect to Φ , denoted $\iota : \mathcal{C} \hookrightarrow \Phi(\mathcal{C})$, is uniquely characterized by the property that composition with ι induces an equivalence $\Phi\text{-Cocts}[\Phi(\mathcal{C}), \mathcal{D}] \cong [\mathcal{C}, \mathcal{D}]$, whose inverse is given by left Kan extension along ι .

⁴ A functor $\mathbf{J} : \mathcal{A} \rightarrow \mathcal{C}$ is *dense* if the left Kan extension of J along itself is (naturally isomorphic to) the identity functor on \mathcal{C} ; that is, each X of \mathcal{C} can be expressed as a canonical colimit $\mathcal{C}(J-, X) * J$ [23, Chapter 5].

Kan extension. Then, in Section 4.B, we characterize among all functors $\mathbf{Pos} \rightarrow \mathbf{Pos}$ those functors that are posetifications.

4.A. Posetifications and functors $\mathbf{Pos} \rightarrow \mathbf{Pos}$ definable in discrete arities.

In order to relate \mathbf{Set} and \mathbf{Pos} -functors, we give below an improved version of [6, Definition 1]:

Definition 4.1. Let T be an endofunctor on \mathbf{Set} . An endofunctor $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ is said to be a *Pos-extension* of T if T' is locally monotone and if the square

$$(14) \quad \begin{array}{ccc} \mathbf{Pos} & \xrightarrow{T'} & \mathbf{Pos} \\ \uparrow D & \lrcorner \alpha & \uparrow D \\ \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \end{array}$$

commutes up to a natural isomorphism $\alpha : DT \rightarrow T'D$.

A \mathbf{Pos} -extension T' is called the *posetification* of T if the above square exhibits T' as $\mathbf{Lan}_D(DT)$ (in the \mathbf{Pos} -enriched sense), having α as its unit.

Intuitively, an extension will coincide with T on discrete sets. One would be tempted to take $T' = DTV$ as an extension of T ; but this is not necessarily locally monotone, as V fails to be so. There is also the possibility of choosing $T' = DTC$, which does produce an extension, but not the posetification.⁵

Remark 4.2.

- (1) Extensions are not necessarily unique. For example, the identity functor on \mathbf{Pos} obviously extends the identity functor on \mathbf{Set} , but the same does the functor DC sending a poset to the (discrete) set of its connected components.
- (2) In general extensions do not need to inherit all the properties of the \mathbf{Set} -functors that they extend. For example, extensions of finitary functors are not necessarily finitary [6, Footnote 2]: consider the (finitary) functor on \mathbf{Set} which maps a set X to the set of almost constant sequences on X ,

$$TX = \{l : \mathbb{N} \rightarrow X \mid l(n) = l(n+1) \text{ for all but a finite number of } n\}$$

It admits the \mathbf{Pos} -extension

$$T'(X, \leq) = \{l : (\mathbb{N}, \leq) \rightarrow (X, \leq) \mid l(n) \leq l(n+1) \text{ for all but a finite number of } n\}$$

mapping a poset (X, \leq) to the poset of almost monotone sequences on X , ordered component-wise. But this extension T' is not finitary: to see this, consider the family of finite posets $\mathfrak{n} = \{0, \dots, n-1\}$ with the usual order, with inclusion maps, whose colimit in \mathbf{Pos} is (\mathbb{N}, \leq) . Then one can easily check that T' does not preserve the above colimit.

It is clear from general considerations that every *finitary* \mathbf{Set} -endofunctor has a posetification. The point of the next theorem is to drop that restriction.

Theorem 4.3. *Each \mathbf{Set} -endofunctor has a posetification.*

⁵ In fact, DTC is the *right* Kan extension $\mathbf{Ran}_D DT$.

$$DX_1 \xrightleftharpoons[Dd_1]{Dd_0} DX_0 \xrightarrow{c_X} X$$
$$DTX_1 \xrightleftharpoons[DTd_1]{DTd_0} DTX_0 \xrightarrow{ex} T'X$$

\curvearrowright
 DTi

(1) Consider a monotone map $f : X \rightarrow Y$. It induces the obvious maps $f_0 : X_0 \rightarrow Y_0$ and $f_1 : X_1 \rightarrow Y_1$. Moreover, the squares

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ d_0 \downarrow & & \downarrow d_0 \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ d_1 \downarrow & & \downarrow d_1 \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

$$e_Y \circ DTf_0 \circ DTd_0 \leq e_Y \circ DTf_0 \circ DTd_1$$

(3) We show that T' is locally monotone; that is, $T'f \leq T'g$ whenever $f \leq g$ holds, for monotone maps $f, g : X \rightarrow Y$. Observe that $f \leq g$ yields a map $\tau : X_0 \rightarrow Y_1$, $x \mapsto (f(x), g(x))$, such that the triangles

$$\begin{array}{ccc} X_0 & \xrightarrow{\tau} & Y_1 \\ & \searrow f_0 & \downarrow d_0 \\ & & Y_0 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\tau} & Y_1 \\ & \searrow g_0 & \downarrow d_1 \\ & & Y_0 \end{array}$$

$$T'g \circ e_X = e_Y \circ DTg_0 = e_Y \circ DTd_1 \circ DT\tau,$$

and from the fact that $e_Y \circ DTd_0 \leq e_Y \circ DTd_1$ holds.

- (4) To prove $T' \cong \text{Lan}_D(DT)$, we shall show that there is an isomorphism between the poset of natural transformations $T' \rightarrow H$ and the poset of natural transformations $DT \rightarrow HD$, for every $H : \text{Pos} \rightarrow \text{Pos}$ (see Remark 3.5(1)).
- (a) Consider a natural transformation $\alpha : DT \rightarrow HD$. For every poset X , we define $\check{\alpha}_X : T'X \rightarrow HX$ as the unique mediating map out of a coinserter:

$$\begin{array}{ccc} DTX_0 & \xrightarrow{e_X} & T'X \\ \alpha_{X_0} \downarrow & & \downarrow \check{\alpha}_X \\ HDX_0 & \xrightarrow{Hc_X} & HX \end{array}$$

Recall that, above, $c_X : DX_0 \rightarrow X$ is a coinserter of Dd_0, Dd_1 . The above definition makes sense since

$$Hc_X \circ \alpha_{X_0} \circ DTd_0 \leq Hc_X \circ \alpha_{X_0} \circ DTd_1$$

holds: the equalities $\alpha_{X_0} \circ DTd_0 = HDd_0 \circ \alpha_{X_1}$ and $\alpha_{X_0} \circ DTd_1 = HDd_1 \circ \alpha_{X_1}$ follow by naturality and $c_X \circ d_0 \leq c_X \circ d_1$ holds, since c_X is a coinserter.

We prove that $\check{\alpha}$ is natural. Consider any monotone map $f : X \rightarrow Y$ and compare

$$\begin{array}{ccc} DTX_0 & \xrightarrow{e_X} & T'X \\ DTf_0 \downarrow & & \downarrow T'f \\ DTY_0 & \xrightarrow{e_Y} & T'Y \\ \alpha_{Y_0} \downarrow & & \downarrow \check{\alpha}_Y \\ HDY_0 & \xrightarrow{Hc_Y} & HY \end{array}$$

with

$$\begin{array}{ccc} DTX_0 & \xrightarrow{e_X} & T'X \\ \alpha_{X_0} \downarrow & & \downarrow \check{\alpha}_X \\ HDX_0 & \xrightarrow{Hc_X} & HX \\ HDf_0 \downarrow & & \downarrow Hf \\ HDY_0 & \xrightarrow{Hc_Y} & HY \end{array}$$

Using naturality of α and co-universality of e_X , we conclude $Hf \circ \check{\alpha}_X = \check{\alpha}_Y \circ T'f$.

- (b) Given a natural transformation $\beta : T' \rightarrow H$, we define, for every set X_0 , the mapping $\hat{\beta}_{X_0} : DTX_0 \rightarrow HDX_0$ to be $\beta_{DX_0} : T'DX_0 \rightarrow HDX_0$ (Here we have used the fact that $T'DX_0$ is naturally isomorphic to DTX_0).
- (c) It is easy then to see that the assignments $\alpha \mapsto \check{\alpha}$ and $\beta \mapsto \hat{\beta}$ are monotone and inverse to each other.

□

As a corollary of the proof of the above theorem (replace DT by T everywhere) we obtain

Corollary 4.4. *For every $T : \mathbf{Set} \rightarrow \mathbf{Pos}$, the enriched left Kan extension $\text{Lan}_D T : \mathbf{Pos} \rightarrow \mathbf{Pos}$ does exist.*

Remark 4.5 (Presentations by monotone operations and equations in discrete arities).

- (1) The posetification built in Theorem 4.3 coincides with the one from [6, (3.2)] given by the coequalizer in \mathbf{Pos}

$$(15) \quad \coprod_{m, n < \omega} \mathbf{Set}(m, n) \times Tm \times [Dn, X] \rightrightarrows \coprod_{n < \omega} Tn \times [Dn, X] \rightarrow \text{Lan}_D(DT)(X)$$

if T is finitary (this follows from the fact that $\text{Lan}_D(DT) \cong \text{Lan}_{DI}(DTI)$, where $I : \mathbf{Set}_f \rightarrow \mathbf{Set}$ is the inclusion).

- (2) Let us explain how (15) gives a *presentation by monotone operations and equations in discrete arities*. The operations of arity n are given by Tn . They are necessarily monotone because the arguments $[Dn, X]$ form a poset and we take the coequalizer in \mathbf{Pos} . The arities are discrete because m, n range over sets, not posets. For each pair (m, n) , we have a poset of equations $\mathbf{Set}(m, n) \times Tm \times [Dn, X]$ (where the order on the equations does not play a role in the computation of the coequalizer).
- (3) For an explicit example of such a presentation by operations and equations, consider T to be the finite powerset functor. First, recall that it can be presented in \mathbf{Set} as the quotient of $\coprod_{n < \omega} X^n$ by a set of equations specifying that the order and the multiplicity in which elements of the set X occur in lists in X^n does not matter. Second, with X now standing for a poset, note that according to [6, Proposition 5], we obtain the posetification of T by quotienting $\coprod_{n < \omega} [Dn, X]$ in \mathbf{Pos} by the same equations. It is not difficult to show that this gives us the (finite) convex powerset functor on \mathbf{Pos} [30, Proposition 5.1].
- (4) If we generalize from the posetification of a finitary functor $\mathbf{Set} \rightarrow \mathbf{Set}$ to the left Kan extension of a finitary functor $\mathbf{Set} \rightarrow \mathbf{Pos}$ the formula (15) is still available and we obtain the same presentations as in item (2), just that the Tn need not be discrete anymore. For example, if we let the Tn in (15) be $\mathcal{P}(n)$ ordered by inclusion, we get a presentation of the functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ mapping a poset X to the set of finitely generated down sets ordered by inclusion (Hoare powerdomain).
- (5) If we generalize further, giving up that the functor be finitary, we loose the formula (15) since the coproducts may not exist in \mathbf{Pos} . Nevertheless, we can still interpret a functor $T : \mathbf{Set} \rightarrow \mathbf{Pos}$ as a *presentation by monotone operations and equations in discrete arities*. This time the arities range over all cardinals, so that for each cardinal \aleph we have a poset of operations $T(\aleph)$ and for each pair of cardinals (\aleph, \aleph') we have a set of equations $\mathbf{Set}(\aleph, \aleph') \times T(\aleph) \times [D(\aleph'), X]$.

The next result is the \mathbf{Pos} -enriched version of [20, Lemma 0.17] (see also [27, Section 2]).

Lemma 4.6. (*3 × 3 lemma for coinserter*) Consider in a Pos-category \mathcal{C} the diagram

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_3} & X_3 \\
 a_1 \downarrow & a_2 & b_1 \downarrow & b_2 & c_1 \downarrow c_2 \\
 Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_3} & Y_3 \\
 a_3 \downarrow & a_2 & b_3 \downarrow & b_2 & c_3 \downarrow \\
 Z_1 & \xrightarrow{h_1} & Z_2 & \xrightarrow{h_3} & Z_3 \\
 & h_2 & & &
 \end{array}$$

where

- (1) The first two rows and columns are coinserter.
- (2) The equalities below hold:

$$(16) \quad b_i \circ f_j = g_j \circ a_i \quad (i, j = 1, 2)$$

These induce the arrows c_1, c_2, h_1, h_2 in an obvious way. Finally, let h_3 be the coinserter (assuming it exists in \mathcal{C}) of h_1 and h_2 , and denote by $c_3 : Y_3 \rightarrow Z_3$ the induced unique mediating arrow. Then:

- (1) The last column is also a coinserter.
- (2) If additionally the first row and columns are reflexive coinserter, then the diagonal

$$X_1 \xrightarrow[b_2 \circ f_2]{b_1 \circ f_1} Y_2 \xrightarrow{h_3 \circ b_3} Z_3$$

is again a coinserter, which is reflexive if the second row (column) is again a reflexive coinserter.

- (3) Reflexivity of the first two rows and columns imply reflexivity of the remaining third row and column.

Proof. To see that c_3 is a coinserter, use first the 2-dimensional aspect of the coinserter (X_3, f_3) to conclude $c_3 \circ c_1 \leq c_3 \circ c_2$. Next, given $w_1 : Y_3 \rightarrow W$ with $w_1 \circ c_1 \leq w_1 \circ c_2$, it induces an arrow $w_2 : Z_2 \rightarrow W$ such that $w_2 \circ b_3 = w_1 \circ g_3$. Then the 2-dimensional part of the coinserter (Z_1, a_3) yields $w_2 \circ h_1 \leq w_2 \circ h_2$, thus it induces an arrow $w_3 : Z_3 \rightarrow W$ with $w_3 \circ h_3 = w_2$. We have that

$$w_3 \circ c_3 \circ g_3 = w_3 \circ h_3 \circ b_3 = w_2 \circ b_3 = w_1 \circ g_3$$

and using that g_3 is an epimorphism we conclude $w_3 \circ c_3 = w_1$. Finally, if $w_1, \bar{w}_1 : Y_3 \rightarrow W$ are such that $w_1 \leq \bar{w}_1$, $w_1 \circ c_1 \leq w_1 \circ c_2$ and $\bar{w}_1 \circ c_1 \leq \bar{w}_1 \circ c_2$, then successively we obtain $w_2 \leq \bar{w}_2$ and $w_3 \leq \bar{w}_3$ by using the 2-dimensional aspect of coinserter (Z_2, b_3) , respectively (Z_3, h_3) .

For the second part, denote by $i : X_2 \rightarrow X_1$ and $j : Y_1 \rightarrow X_1$ the common right inverses of the parallel pairs of morphisms f_1, f_2 , respectively a_1, a_2 . Notice then that for an arrow $u_1 : Y_2 \rightarrow U$ such that $u_1 \circ b_1 \circ f_1 \leq u_1 \circ b_2 \circ f_2$, precomposition with i induces the inequality $u_1 \circ b_1 \leq u_1 \circ b_2$, thus we can find an arrow $u_2 : Z_2 \rightarrow U$ with $u_2 \circ b_3 = u_1$. In order to see that $u_2 \circ h_1 \leq u_2 \circ h_2$, use the first that precomposing

$$u_1 \circ g_1 \circ a_1 = u_1 \circ b_1 \circ f_1 \leq u_1 \circ b_2 \circ f_2 = u_1 \circ g_2 \circ a_2$$

with j yields $u_1 \circ g_1 \leq u_1 \circ g_2$, and next use the 2-dimensional aspect of the coinserter (Z_1, a_3) . From $u_2 \circ h_1 \leq u_2 \circ h_2$ we see that there is an arrow $u_3 : Z_3 \rightarrow U$ with

$u_3 \circ h_3 = u_2$, thus $u_3 \circ h_3 \circ b_3 = u_2 \circ b_3 = u_1$. The remaining 2-dimensional aspect of the requested coinserters can be easily proved along these lines, and we leave it to the reader, as well as the assertions on reflexivity. \square

Proposition 4.7. *Posetifications of Set functors preserve reflexive coinserters in Pos.*

Proof. It follows from the above lemma and from the construction of posetifications as reflexive coinserters. \square

Corollary 4.8. *Taking posetifications commutes with functor composition. That is, given two functors $T, S : \mathbf{Set} \rightarrow \mathbf{Set}$, there is a (canonical) isomorphism of locally monotone Pos functors*

$$S'T' \cong (ST)'$$

Proof. Consider an arbitrary poset X and express it as a coinserters of its truncated nerve. Then $T'X$ is the coinserters of the reflexive pair $DTd_0, DTd_1 : DTX_1 \rightarrow DTX_0$, and S' preserves such weighted colimits by the above proposition. Henceforth the conclusion follows. \square

Remark 4.9 (On posetifications and relation lifting).

- (1) Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ be an arbitrary functor. For a relation $\mathbf{r} \subseteq X \times Y$, recall that the T -relation lifting of \mathbf{r} is (see for example [8, 16, 39, 40]):

$$\begin{array}{ccc} & \mathbf{r} & \\ \pi_1 \swarrow & \downarrow & \searrow \pi_2 \\ X & X \times Y & Y \end{array} \quad \begin{array}{ccc} & T\mathbf{r} & \\ T\pi_1 \swarrow & \downarrow \text{Rel}_T(\mathbf{r}) & \searrow T\pi_2 \\ TX & TX \times TY & TY \end{array}$$

Explicitly,

$$\mathbf{Rel}_T(\mathbf{r}) = \{(u, v) \in TX \times TY \mid \exists w \in T\mathbf{r} . T\pi_1(w) = u \wedge T\pi_2(w) = v\}$$

The relation lifting satisfies the following properties:

- (a) It preserves the equality relation: $=_{TX} = \mathbf{Rel}_T(=_X)$.
- (b) It preserves the inclusion of relations: if $\mathbf{r} \subseteq \mathbf{s}$ then $\mathbf{Rel}_T(\mathbf{r}) \subseteq \mathbf{Rel}_T(\mathbf{s})$.
- (c) If $\mathbf{r} \subseteq X \times Y$ and $\mathbf{s} \subseteq Y \times Z$, then $\mathbf{Rel}_T(\mathbf{s} \circ \mathbf{r}) \subseteq \mathbf{Rel}_T(\mathbf{s}) \circ \mathbf{Rel}_T(\mathbf{r})$, with equality if and only if T preserves weak pullbacks.
- (d) It preserves converses of relations: $\mathbf{Rel}_T(\mathbf{r}^{\text{op}}) = \mathbf{Rel}_T(\mathbf{r})^{\text{op}}$.
- (e) Given functions $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ and relation $\mathbf{r}' \subseteq X' \times Y'$, then

$$\mathbf{Rel}_T((f \times g)^{-1}(\mathbf{r}')) \subseteq (Tf \times Tg)^{-1}(\mathbf{Rel}_T(\mathbf{r}'))$$

with equality if T preserves weak pullbacks.

- (2) In addition to the above, we should also mention the (less-known?) fact that relation lifting commutes with functor composition, in the sense that

$$\mathbf{Rel}_{T \circ S}(\mathbf{r}) = \mathbf{Rel}_T(\mathbf{Rel}_S(\mathbf{r}))$$

for any relation $\mathbf{r} \subseteq X \times Y$ and any Set-functors T, S (see [16, Section 4.4], and use that any Set-functor preserves strong epimorphisms, i.e. surjective maps, assuming the axiom of choice).

- (3) Recall again that the posetification T' of a **Set**-endofunctor T was obtained via coinserter,

$$DTX_1 \xrightarrow[DTd_1]{DTd_0} DTX_0 \longrightarrow T'X$$

for any poset X . Observe in fact that the relation \mathbf{r} described in Remark 3.2 at the first stage of the inserter construction, for the pair of (monotone) maps DTd_0 and DTd_1 , is precisely the transitive closure of the T -relation lifting $\mathbf{Rel}_T(X_1)$ of the order X_1 on X .⁶ By (1a) above, $\mathbf{Rel}_T(X_1)$ is reflexive, and by (1c) it is also transitive if T preserves weak pullbacks. If this is the case, then the posetification T' can be explicitly described as mapping a poset X to the quotient poset of the preordered set $(TX, \mathbf{Rel}_T(X_1))$.⁷

- (4) The above two items provide a new proof of the result in Corollary 4.8, but only for weak-pullbacks preserving functors.

Definition 4.10 ([19]). An *exact square* in the category **Pos** of posets, or in the category **Preord** of preorders, is a diagram

$$(17) \quad \begin{array}{ccc} E & \xrightarrow{\alpha} & X \\ \beta \downarrow & \swarrow & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

with $f \circ \alpha \leq g \circ \beta$, such that

$$(18) \quad \forall x \in X, y \in Y. f(x) \leq g(y) \Rightarrow \exists w \in E. (x \leq \alpha(w) \wedge \beta(w) \leq y)$$

The reader should think of an exact square as being the **Pos**-enriched analogue of a weak pullback. In fact, an exact square of discrete posets is precisely a weak pullback of their underlying sets. Equivalently, the discrete functor D maps weak pullbacks to exact squares and reflects exact squares to weak pullbacks.

Given a **Set**-functor T , we shall now connect the property of preserving weak pullbacks with the preservation of exact squares by the corresponding posetification T' . This will be used later in the paper (Theorem 6.12).

Proposition 4.11. *Let T be any **Set**-functor and T' its posetification. Then T preserves weak pullbacks if and only if T' preserves exact squares.*

Proof. This was proved in [6] under the additional assumption that T is finitary. Here, we present an argument valid for all **Set**-functors.

We start with the easy implication. Assume T' preserves exact squares and consider a weak pullback in **Set**

$$(19) \quad \begin{array}{ccc} E & \xrightarrow{\alpha} & X \\ \beta \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

⁶We identify the order relation with the set of comparable pairs.

⁷In case T is also finitary, this was noticed in [6, Proposition 13].

Then (19) is mapped by D to an exact square in \mathbf{Pos} , and T' preserves such by hypothesis. Using the isomorphism $DT \cong T'D$, we conclude that

$$\begin{array}{ccc} DTE & \xrightarrow{DT\alpha} & DTX \\ DT\beta \downarrow & & \downarrow DTf \\ DTY & \xrightarrow{DTg} & Z \end{array}$$

is an exact square of discrete posets, that is, a weak pullback in \mathbf{Set} .

Now, we assume that T preserves weak pullbacks and we show that its posetification T' preserves exact squares. Remember from Remark 4.9(3) that the property of weak pullbacks preservation for T entails that on a poset X , $T'X$ is the quotient of the preordered set $(TX_0, \text{Rel}_T(X_1))$.

In fact, it is easy to see that for each preordered set (poset) X , the construct

$$\mathbf{Preord}(T)X = (TX_0, \text{Rel}_T(X_1))$$

yields a locally monotone functor $\mathbf{Preord}(T)$ on the category \mathbf{Preord} of preordered sets and monotone mappings.⁸

The inclusion functor $\text{Incl} : \mathbf{Pos} \rightarrow \mathbf{Preord}$ and its left adjoint, the quotient functor $\text{Quot} : \mathbf{Preord} \rightarrow \mathbf{Pos}$ both preserve exact squares [9, Example 6.2], and the composite

$$\mathbf{Pos} \xrightarrow{\text{Incl}} \mathbf{Preord} \xrightarrow{\mathbf{Preord}(T)} \mathbf{Preord} \xrightarrow{\text{Quot}} \mathbf{Pos}$$

is precisely T' . Consequently, it is enough to show that $\mathbf{Preord}(T)$ preserves exact squares.

Consider thus an exact square in \mathbf{Preord} :

$$(20) \quad \begin{array}{ccc} E & \xrightarrow{\alpha} & X \\ \beta \downarrow & \swarrow & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

and follow the steps below:

- (1) First, the inequality $\mathbf{Preord}(T)(f) \circ \mathbf{Preord}(T)(\alpha) \leq \mathbf{Preord}(T)(g) \circ \mathbf{Preord}(T)(\beta)$ holds by the local monotonicity of $\mathbf{Preord}(T)$.
- (2) Next, we consider a diagram with three pullbacks in \mathbf{Set} , see below, which by hypothesis will be mapped by T to weak pullbacks. To avoid overloaded notation, we shall slightly abuse and denote by same symbol both the preordered set and its underlying set, and do the same for (monotone) mappings between them. As in Proposition 3.3, let $d_0, d_1 : Z_1 \rightarrow Z$ stand for the projections from the set of comparable pairs to (the underlying set of) Z .

⁸In [6], this was noticed only for finitary \mathbf{Set} -functors T , $\mathbf{Preord}(T)$ being called the *preodification* of T .

$$(21) \quad \begin{array}{ccc} & R & \\ r_0 \swarrow & & \searrow r_1 \\ P & & Q \\ p_1 \searrow & Z_1 & \swarrow q_0 \\ p_0 \downarrow & & \downarrow q_1 \\ X & & Y \\ f \searrow & Z & \swarrow g \end{array} \Rightarrow \begin{array}{ccc} & TR & \\ Tr_0 \swarrow & & \searrow Tr_1 \\ TP & & TQ \\ Tp_1 \searrow & TZ_1 & \swarrow Tq_0 \\ Tp_0 \downarrow & & \downarrow Tq_1 \\ X & & TY \\ Tf \searrow & Z & \swarrow Tg \end{array}$$

Explicitly,

$$\begin{aligned} P &= \{(x, z) \in X \times Z \mid f(x) \leq z\} \\ Q &= \{(z, y) \in Z \times Y \mid z \leq g(y)\} \\ R &= \{(x, y) \in X \times Y \mid f(x) \leq g(y)\} \\ r_0(x, y) &= (x, g(y)) \quad r_1(x, y) = (f(x), y) \end{aligned}$$

- (3) From the description of R above, notice that R is non-empty (as we started from an exact square), and that given $(x, y) \in R$, there is some $w \in E$ such that $x \leq \alpha(w)$ and $\beta(w) \leq y$. Assuming the axiom of choice, fix such an $w \in E$ for each $(x, y) \in R$ and define a map $\theta : R \rightarrow E$ by $\theta(x, y) = w$. It can be considered monotone if R is taken to be a discrete poset.
- (4) Consider the cube below in \mathbf{Preord} , where R, P, Q carry the discrete (pre)order.

$$\begin{array}{ccccc} R & \xrightarrow{r_0} & P & & \\ & \searrow \theta & & & \\ & & E & \xrightarrow{\alpha} & X \\ r_1 \downarrow & & \downarrow p_1 & & \downarrow f \\ Q & \xrightarrow{q_0} & Z_1 & & \\ & \searrow q_1 & & & \\ & & Y & \xrightarrow{g} & Z \end{array}$$

The back, right and bottom faces commute from (21). The front face is the exact square of (20), in particular $f \circ \alpha \leq g \circ \beta$ holds. The remaining up and left faces laxly commute, in the sense that following inequalities hold:

$$(22) \quad p_0 \circ r_0 \leq \alpha \circ \theta \quad \text{and} \quad \beta \circ \theta \leq q_1 \circ r_1$$

- (5) We are now able to show that $\mathbf{Preord}(T)$ preserves exact squares. Let thus $u \in \mathbf{Preord}(T)(X)$, $v \in \mathbf{Preord}(T)(Y)$ such that

$$\mathbf{Preord}(T)(f)(u) \leq \mathbf{Preord}(T)(g)(v)$$

in $\mathbf{Preord}(T)(Z) = (TZ, \text{Rel}_T(Z_1))$. That is, $u \in TX$, $v \in TY$ and there exists some $w \in T(Z_1)$ such that $Td_0(w) = Tf(u)$ and $Td_1(w) = Tg(v)$.

As all the squares in the second diagram in Equation (21) are weak pullbacks, we can conclude that there is some $\bar{w} \in TR$ which is mapped to $u \in TX$, respectively $v \in TY$. Let $\omega = T\theta(\bar{w}) \in TE$. Then one can easily check using Equation (22) that $u \leq T\alpha(\omega)$ and $T\beta(\omega) \leq v$ hold.

All in one, we have showed that $\mathbf{Preord}(T)$ maps an exact square to an exact square. Thus also the posetification of T preserves exact squares. \square

Example 4.12.

- (1) Let $T = \text{Id}$ on \mathbf{Set} . Then its posetification is the identity functor on posets (recall that the discrete-poset functor D is dense, see the last paragraph in Section 2 of [15]).
- (2) If we take $T = \mathcal{P}_f$ to be the (finite) power-set functor, then its posetification is the (finitely generated) convex power-set functor, with the Egli-Milner order [6, 30].
- (3) The collection of (finitary) Kripke polynomial \mathbf{Set} -functors is inductively defined as follows: $T ::= \text{Id} \mid T_{X_0} \mid T_0 + T_1 \mid T_0 \times T_1 \mid T^A \mid \mathcal{P}_f$, where T_{X_0} denotes the constant functor to the set X_0 ; $T_0 + T_1$ is the coproduct functor $X \mapsto T_0X + T_1X$; $T_0 \times T_1$ the product functor; and T^A denotes the exponent functor $X \mapsto (TX)^A$, with A finite.

We have just said above that the posetification of the identity functor is again the identity, while for the constant functor T_{X_0} it is an easy exercise to check that the posetification is again a constant functor, this time to the discrete poset DX_0 ; the posetification of the coproduct functor $T_0 + T_1$ maps a poset X to the coproduct (in the category of posets) $T'_0X + T'_1X$, where T'_0 and T'_1 denote the posetifications of T_0 , respectively T_1 ; and similarly for the product and exponent functors.

- (4) Consider now the finitary probability functor $\mathbf{Prob} : \mathbf{Set} \rightarrow \mathbf{Set}$, given by

$$\mathbf{Prob}X = \{p : X \rightarrow [0, 1] \mid \sum_{x \in X} p(x) = 1, \text{ supp}(p) < \infty\}^9$$

and

$$\mathbf{Prob}(f)(p)(y) = \sum_{y=f(x)} p(x), \text{ for a function } f : X \rightarrow Y.$$

Recall that \mathbf{Prob} preserves weak pullbacks [41], thus its posetification \mathbf{Prob}' can be described using the relation lifting as in Rem. 4.9. In fact, for the probability functor, it happens that the relation lifting of a partial order is not just a preorder, but even a partial order [5]. Henceforth for a poset X , $\mathbf{Prob}'X$ has the underlying set $\mathbf{Prob}X_0$, ordered by the following: for $p, p' \in \mathbf{Prob}X_0$, $p \leq p'$ iff there is some $\omega \in \mathbf{Prob}(X_0 \times X_0)$ such that $\sum_{x' \in X} \omega(x, x') = p(x)$ and $\sum_{x \in X} \omega(x, x') = p'(x')$, and $\omega(x, x') > 0 \Rightarrow x \leq x'$.

4.B. Characterising functors $\mathbf{Pos} \rightarrow \mathbf{Pos}$ in discrete arities.

Recall from Proposition 3.3 that we have denoted, for each poset X , by N_X the diagram

$$DX_1 \begin{array}{c} \xrightarrow{Dd_1} \\ \xrightarrow{Dd_0} \end{array} DX_0$$

of discrete posets, where X_0 is the set of elements of X and X_1 is the set of all pairs (x', x) with $x' \leq x$ in X . The maps d_0, d_1 are the obvious projections.

⁹The support of a probability function $p : X \rightarrow [0, 1]$ is $\text{supp}(p) = \{x \in X \mid p(x) \neq 0\}$.

Theorem 4.13. *For $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$, the following are equivalent:*

- (1) *There exists a functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ such that $T' \cong \text{Lan}_D(DT)$, i.e., T' is a posetification of T .*
- (2) *T' preserves discrete posets and coinserter of all diagrams N_X .*

Proof. We prove first that the coinserter of diagrams N_X form the density presentation of $D : \mathbf{Set} \rightarrow \mathbf{Pos}$ in the sense of [23, Section 5.4]. Indeed, all coinserter of N_X exist in \mathbf{Pos} , the category \mathbf{Pos} is the closure of \mathbf{Set} under these coinserter, and the coinserter of N_X are preserved by the functor $\mathbf{Pos}(D-, -) : \mathbf{Pos} \rightarrow [\mathbf{Set}^{\text{op}}, \mathbf{Pos}]$. To see the latter, observe that for any set S , the poset X^S is a coinserter of

$$(23) \quad (X_1)^S \xrightleftharpoons[(d_0)^S]{(d_1)^S} (X_0)^S$$

We prove now that (1) implies (2). Since $TD \cong DT'$ holds, T preserves discrete posets. By (23), the collection of all coinserter of N_X forms a density presentation of D , hence by [23, Theorem 5.29], T preserves coinserter of all diagrams in N_X .

(2) implies (1). Since T' is assumed to preserve discrete posets, we may assume that $T'D \cong DT$ for some functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$. Furthermore, by [23, Theorem 5.29], $T' \cong \text{Lan}_D(T'D)$ holds. That is, $T' \cong \text{Lan}_D(DT)$ holds. \square

Above, one can drop the requirement that T preserves discrete posets:

Corollary 4.14. *A functor $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ is of the form $\text{Lan}_D S$ for some $S : \mathbf{Set} \rightarrow \mathbf{Pos}$ iff T' preserves coinserter of all diagrams N_X .*

Similarly, one has:

Theorem 4.15. *A functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ is strongly finitary and preserves discrete posets if and only if it is the posetification of a finitary functor $\mathbf{Set} \rightarrow \mathbf{Set}$. A functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ is strongly finitary if and only if it is the left Kan extension of a finitary functor $\mathbf{Set} \rightarrow \mathbf{Pos}$.*

5. PRESENTING FUNCTORS ON ORDERED VARIETIES BY OPERATIONS AND EQUATIONS

Coming back to the introduction, we remind the reader that our overall strategy is, starting with a functor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ giving the type of coalgebras, to obtain from T the Boolean logic $L : \mathbf{BA} \rightarrow \mathbf{BA}$ by Stone duality and to obtain from the posetification T' of T , again by duality, the positive logic $L' : \mathbf{DL} \rightarrow \mathbf{DL}$. The relationship between L and L' will be studied in the next section. Here, we are going to make sure that the functors L and L' obtained by abstract categorical constructions actually do have concrete presentations by operations and equations and thus correspond indeed to modal extensions of Boolean and positive propositional logic.

In the case of L , assuming that L preserves sifted colimits, this is known already from [29] and will be recalled below. In the case of L' , we need to prove the enriched analogue of [29] which we shall obtain following the enriched generalization of [29] given in [30]. In particular, this enriched generalization will guarantee that L' can be presented by *monotone* operations. As a final twist, this enriched generalization would give us a presentation of L' using inequations over general ordered varieties. Therefore, it is important for us to show that, owing to the special nature of \mathbf{DL} ,

the enriched functor L' can equally be presented as the underlying ordinary functor L'_o , which in turn has a presentation that does not rely on inequations.

5.A. Equational presentation of functors.

We have seen, in Example 2.1, a presentation of a functor $L : \mathbf{BA} \rightarrow \mathbf{BA}$ and in Example 2.6 a presentation of a functor $L' : \mathbf{DL} \rightarrow \mathbf{DL}$. Whereas it may be clear from these examples what we mean by a presentation, it is worth spending the effort to give a formal definition.

In what follows, \mathcal{A} will denote a variety of algebras for a finitary signature. By a slight abuse of notation, we shall use the same notation as in case of the variety \mathbf{BA} for the (monadic) adjunction

$$F \dashv U : \mathcal{A} \rightarrow \mathbf{Set}$$

We will use Σ_n to denote the set of n -ary modal operators and Γ_n to denote the set of equations in n free variables. For instance, for Example 2.1 we have $\Sigma_1 = \{\Box\}$ and $\Sigma_n = \emptyset$ for $n \neq 1$, and $\Gamma_0 = \{(\Box\top, \top)\}$ and $\Gamma_2 = \{(\Box(a \wedge b), \Box a \wedge \Box b)\}$ and $\Gamma_m = \emptyset$ for $m \neq 0, 2$. Given a signature $\Sigma = (\Sigma_n)_{n < \omega}$, we write $\hat{\Sigma} : \mathbf{Set} \rightarrow \mathbf{Set}$ for the corresponding polynomial functor $X \mapsto \coprod_{n < \omega} \mathbf{Set}(n, X) \bullet \Sigma_n$. Observe that with this notation, in Example 2.1 we have that $\Gamma_n \subseteq UF\hat{\Sigma}UFn \times UF\hat{\Sigma}UFn$ (interpret $UF\hat{\Sigma}$ as the set of Boolean terms on X -generators).

Definition 5.1. [13, Definition 6] A functor L on an variety \mathcal{A} has a *presentation by operations and equations*, or, shortly, a *presentation*, if there are signatures Σ and Γ , with $\Gamma_n \subseteq UF\hat{\Sigma}UFn \times UF\hat{\Sigma}UFn$, such that for all $A \in \mathcal{A}$ the following diagram, where n ranges over natural numbers and v ranges over all valuations $Fn \rightarrow A$ (of n -variables in A)

$$F\hat{\Gamma}n \rightrightarrows F\hat{\Sigma}UFn \xrightarrow{F\hat{\Sigma}Uv} F\hat{\Sigma}UA \longrightarrow LA$$

is a joint coequalizer.

Remark 5.2 (Axioms of rank 1). We see that the format of the equations (i.e. the elements of the Γ_n) requires them to be pairs in $UF\hat{\Sigma}UFn \times UF\hat{\Sigma}UFn$, that is, every variable must be under exactly one modal operator. Such equations are often called *equations of rank 1*. For example, if we wanted to extend \mathbf{DL} by negation (thinking of negation as a unary modal operator), then $\neg(a \wedge b) = \neg a \vee \neg b$ and $\neg 1 = 0$ are of rank 1 one, but $a \wedge \neg a = 0$ is not. The importance of equations of rank 1 is that they are enough to present functors, see the theorem below.

For proofs of the following proposition and theorem see [29, Theorem 4.7].

Proposition 5.3. *A functor L on a variety \mathcal{A} has a presentation iff there are polynomial functors $\hat{\Sigma}, \hat{\Gamma} : \mathbf{Set} \rightarrow \mathbf{Set}$ such that L is a coequalizer*

$$F\hat{\Gamma}U \rightrightarrows F\hat{\Sigma}U \longrightarrow L$$

in the category of endofunctors.

Recall that any ordinary variety \mathcal{A} can be presented by a signature $\Sigma_{\mathcal{A}}$ and equations $E_{\mathcal{A}}$. For instance the variety \mathbf{DL} is presented by the constants \perp, \top , the binary operations \wedge, \vee and the usual equations defining distributive lattices, see eg [17].

Theorem 5.4. *Let L be a functor on a variety \mathcal{A} . Let \mathcal{A} be presented by a signature $\Sigma_{\mathcal{A}}$ and equations $E_{\mathcal{A}}$. Then:*

- (1) *If L has a presentation $\langle \Sigma, \Gamma \rangle$, then the category of L -algebras is isomorphic to the category of algebras for the signature $\Sigma_{\mathcal{A}} + \Sigma$ satisfying the equations $E_{\mathcal{A}}$ and Γ .*
- (2) *A functor L on a variety \mathcal{A} has a presentation iff L preserves ordinary sifted colimits.*

This theorem gives a bijection between sifted colimits preserving functors L on a variety \mathcal{A} and logics extending \mathcal{A} by ‘modal operators’ and axioms of rank 1. The theorem enables us to investigate such logics using purely category theoretic means.

5.B. Equational presentations of locally monotone functors. For the purposes of our investigations, we are interested in modal logics extending DL, given by rank 1 axioms of *monotone* operations. While $U : \mathbf{DL} \rightarrow \mathbf{Set}$ is certainly finitary and monadic (since DL is an ordinary variety of algebras), it is also the case that the natural forgetful functor $U' : \mathbf{DL} \rightarrow \mathbf{Pos}$, mapping a distributive lattice to its carrier equipped with the lattice order, exhibits DL as an *ordered* variety.

For now let us be slightly more general and consider an ordered variety

$$F' \dashv U' : \mathcal{A} \rightarrow \mathbf{Pos}$$

By an *ordered signature* Σ' we shall mean a family of posets $\Sigma' = (\Sigma'_n)_{n < \omega}$. Let $\tilde{\Sigma}' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ be the corresponding polynomial functor $X \mapsto \coprod_{n < \omega} \mathbf{Pos}(Dn, X) \bullet \Sigma'_n$. In the following we shall call a functor $\mathbf{Pos} \rightarrow \mathbf{Pos}$ *polynomial* only if it is of the form $\tilde{\Sigma}$. Notice that a polynomial functor only employs *discrete arities* (if $\Sigma' = D\Sigma$ for some (necessarily unique) Set signature Σ then $\tilde{\Sigma}'$ is the posetification of $\hat{\Sigma}$ in the sense of Definition 4.1).

Definition 5.5. A functor L' on an ordered variety \mathcal{A} has an *ordered presentation in discrete arities*, or, shortly, an *ordered presentation*, if there are ordered signatures Σ' and Γ' , such that for all $A \in \mathcal{A}$, the following diagram, where n ranges over natural numbers and v ranges over all valuations $F'n \rightarrow A$ (of n -variables in A)

$$(24) \quad F'\tilde{\Gamma}'Dn \rightrightarrows F'\tilde{\Sigma}'U'F'Dn \xrightarrow{F'\tilde{\Sigma}'U'v} F'\tilde{\Sigma}'U'A \xrightarrow{q} L'A$$

is a joint coequalizer.

In the definition above it is not important to allow Γ'_n to be posets. On other hand, for general \mathcal{A} it is important to allow the Σ'_n to be posets. Then again, for $\mathcal{A} = \mathbf{DL}$ we can take the Σ'_n discrete since for DL the order is equationally definable, as will be discussed in detail in Section 5.D.

Remark 5.6. An ordered presentation is monotone. In detail, let $\alpha : L'A \rightarrow A$ be an algebra. Consider an operation $\sigma \in \Sigma'_n$ and $a, a' : Dn \rightarrow U'A$ with $a \leq a'$, that is, $a_i \leq a'_i$ for all $1 \leq i \leq n$. We have to show that $\sigma(a) \leq \sigma(a')$ in (A, α) , but this is equivalent to the obvious $U'\alpha \circ q^b(\sigma, a) \leq U'\alpha \circ q^b(\sigma, a')$ where $q^b : \tilde{\Sigma}'U'A \rightarrow UL'A$ is the adjoint transpose of q and (σ, a) and (σ, a') are pairs in $\Sigma'_n \bullet \mathbf{Pos}(Dn, U'A) = \Sigma'_n \times \mathbf{Pos}(Dn, U'A)$.

Proposition 5.7. *A locally monotone functor L' on an ordered variety \mathcal{A} has an ordered presentation iff there are polynomial functors $\tilde{\Sigma}, \tilde{\Gamma} : \mathbf{Pos} \rightarrow \mathbf{Pos}$ such that L is a coequalizer*

$$F'\tilde{\Gamma}U' \rightrightarrows F'\tilde{\Sigma}U' \longrightarrow L'$$

in the category of locally monotone endofunctors.

Theorem 5.8. *Let $L' : \mathcal{A} \rightarrow \mathcal{A}$ be a locally monotone functor on an ordered variety. Then L' preserves Pos-sifted colimits iff it has a ordered presentation in discrete arities.*

Proof. We denote by \mathcal{A}_{ff} the full subcategory of \mathcal{A} spanned by the algebras which are free on finite discrete posets and by $F'_f : \mathbf{Set}_f \rightarrow \mathcal{A}_{ff}$ the domain-codomain restriction of $F'D : \mathbf{Set} \rightarrow \mathbf{Pos} \rightarrow \mathcal{A}$. Then:

- (1) Observe that $[F'_f, U'] : [\mathcal{A}_{ff}, \mathcal{A}] \rightarrow [\mathbf{Set}_f, \mathbf{Pos}]$, sending a functor $L' : \mathcal{A}_{ff} \rightarrow \mathcal{A}$ to the composite $U'L'F'_f$, is of descent type. This follows immediately from [30, Lemma 3.14].
- (2) The functor $[E, -] : [\mathbf{Set}_f, \mathbf{Pos}] \rightarrow [|\mathbf{Set}_f|, \mathbf{Pos}]$, where $|\mathbf{Set}_f|$ is the discrete skeleton of the category of finite sets and $E : |\mathbf{Set}_f| \rightarrow \mathbf{Set}_f$ is the inclusion, is monadic (again, this follows from Lemma 3.14 of [30]).
- (3) Consequently, the composite

$$[\mathcal{A}_{ff}, \mathcal{A}] \xrightarrow{[F'_f, U']} [\mathbf{Set}_f, \mathbf{Pos}] \xrightarrow{[E, -]} [|\mathbf{Set}_f|, \mathbf{Pos}]$$

is of descent type. This follows from [30, Theorem 3.18].

Consequently, every functor $L' : \mathcal{A}_{ff} \rightarrow \mathcal{A}$ (i.e., every L' preserving sifted colimits) has a presentation as in Proposition 5.7. \square

5.C. Ordinary and ordered presentations of functors on BA. Let \mathcal{A} be a \mathbf{Pos} -enriched category with discretely ordered hom-posets, such as BA. Then, as we are going to show now, there is no essential difference in the presentations according to Section 5.B and 5.A.

Before coming to functors on varieties, let us clarify when ordinary varieties are ordered varieties and vice versa. Recall that we wrote $C \dashv D : \mathbf{Set} \rightarrow \mathbf{Pos}$ for the adjunction in which D is the discrete functor and C the connected components functor.

Proposition 5.9. *Let $F' \dashv U' : \mathcal{A} \rightarrow \mathbf{Pos}$ be an ordered variety. It has discretely ordered hom-posets iff any of the following equivalent conditions are satisfied.*

- (1) U' factors through $D : \mathbf{Set} \rightarrow \mathbf{Pos}$.
- (2) $U' \cong DCU'$.
- (3) $\eta_{U'} : U' \rightarrow DCU'$ is an isomorphism, where η is the unit of $C \dashv D$.

If any of the above conditions is satisfied then F' factors through $C : \mathbf{Pos} \rightarrow \mathbf{Set}$ via $F' \cong F'DC$. Moreover, $F'D \dashv CU' : \mathcal{A} \rightarrow \mathbf{Set}$ is monadic.

Conversely, if $F \dashv U : \mathcal{A} \rightarrow \mathbf{Set}$ is a variety and the only order on algebras in \mathcal{A} making all operations monotone is the trivial order (as it is the case in BA), then $FC \dashv DU : \mathcal{A} \rightarrow \mathbf{Pos}$ is an ordered variety, see [32].

Proposition 5.10. *Let $\mathcal{A} \rightarrow \mathbf{Pos}$ be a variety with discretely ordered homsets and $L : \mathcal{A} \rightarrow \mathcal{A}$ be a (necessarily locally monotone) functor. Then the functor L preserves ordinary sifted colimits iff L preserves enriched sifted colimits. Moreover,*

$\langle D\Sigma, D\Gamma \rangle$ is an ordered presentation of L iff $\langle \Sigma, \Gamma \rangle$ is a presentation of L and $\langle \Sigma', \Gamma' \rangle$ is an ordered presentation of L iff $\langle C\Sigma', C\Gamma' \rangle$ is a presentation of L .

The proposition above guarantees that for a functor $L : \mathbf{BA} \rightarrow \mathbf{BA}$, it does not matter whether we consider it an ordinary functor on the variety \mathbf{BA} or whether we consider it as an enriched functor on the ordered variety \mathbf{BA} .

5.D. Ordinary and ordered presentations of functors on DL. The aim of this section is to show that not only a functor on DL has a presentation by operations and equation iff it preserves ordinary sifted colimits, but also that a functor has a presentation by *monotone* operations and equations iff it preserves *enriched* sifted colimits.

We begin with

Proposition 5.11. *If \mathcal{A} is an ordered variety and $L' : \mathcal{A} \rightarrow \mathcal{A}$ preserves enriched sifted colimits, then $L'_o : \mathcal{A}_o \rightarrow \mathcal{A}_o$ preserves ordinary sifted colimits.*

Proof. We know that \mathcal{A} is a free cocompletion by enriched sifted colimits of the full subcategory $I : \mathcal{A}_{ff} \hookrightarrow \mathcal{A}$ spanned by free algebras on a finite and discrete set of generators. Furthermore, $I : \mathcal{A}_{ff} \hookrightarrow \mathcal{A}$ has the following density presentation:

- (1) reflexive coinserter
- (2) (conical) filtered colimits
- (3) reflexive coequalizers

The reason is that we can

- (1) use coinserter of truncated nerves to create algebras, free on any finite poset,
- (2) use (conical) filtered colimits to obtain free algebras on any poset,
- (3) use reflexive coequalizer (=canonical presentation) to obtain any algebra.

Hence, we know that $L : \mathcal{A} \rightarrow \mathcal{A}$ preserves enriched sifted colimits iff L preserves colimits in (1), (2) and (3). Since colimits in (2) and (3) are conical, they are preserved by L_o . But \mathcal{A} was \mathbf{Pos} -cocomplete, hence \mathcal{A}_o is \mathbf{Set} -cocomplete. And a functor between cocomplete categories preserves sifted colimits iff it preserves ordinary filtered colimits and reflexive coequalizers. \square

If \mathcal{A}_o is an ordinary variety, then L' also has a presentation by operations and equations.

Example 5.12. Let $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ be the locally monotone functor presented by one unary operation, written as \square , and no equations. It follows from the proposition that monotonicity of \square is equationally definable. Explicitly, the induced equational presentation of L'_o is given by

$$\square a \wedge \square(a \vee b) = \square a$$

Of course, the proposition only tells us that all finitary equations valid for a monotone \square together present L'_o . But it is not difficult to check that the equation above is enough to force \square to be monotone. \square

Conversely, if \mathcal{A}_o is an ordinary variety, it makes sense to ask how a presentation of a functor on \mathcal{A}_o induces a presentation of a functor on \mathcal{A} . Let $F' \dashv U' : \mathcal{A} \rightarrow \mathbf{Pos}$ be an ordered variety, $\tilde{D} : \mathcal{A}_o \rightarrow \mathcal{A}$ the subcategory which has the same objects

and arrows as \mathcal{A} but discrete homsets, and assume that \mathcal{A}_o is an ordinary variety. The following diagram obviously commutes

$$\begin{array}{ccc} \mathcal{A}_o & \xrightarrow{\tilde{D}} & \mathcal{A} \\ \uparrow F & & \uparrow F' \\ \mathbf{Set} & \xrightarrow{D} & \mathbf{Pos} \end{array}$$

Now, given a presentation $\langle \Sigma, \Gamma \rangle$ of a functor $L_o : \mathcal{A}_o \rightarrow \mathcal{A}_o$ in the sense of Definition 5.1, it induces a presentation $\langle D\Sigma, D\Gamma \rangle$ of a functor $L : \mathcal{A} \rightarrow \mathcal{A}$ in the sense of Definition 5.5. Computing both L_o and L according to (24), we obtain

$$(25) \quad \tilde{D}L_o \rightarrow LD$$

which is necessarily onto, but may fail to be bijective.

Definition 5.13. We say that $\langle \Sigma, \Gamma \rangle$ is a *presentation by monotone operations and equations*, or, shortly, a *monotone presentation*, if (25) is an isomorphism.

This terminology is justified by Remark 5.6, according to which $\langle D\Sigma, D\Gamma \rangle$ is a presentation by monotone operations.

Example 5.14. The presentation of Example 2.6 is a presentation by monotone operations, since to say that \square preserve meets and that \diamond preserves joins forces \square and \diamond to be monotone. On the other hand, if one omitted these axioms from the presentation, one would obtain a presentation that is not monotone.

To summarize, given an ordered presentation $\langle \Sigma', \Gamma' \rangle$ of a functor $L' : \mathcal{A} \rightarrow \mathcal{A}$ on an ordered variety in the sense of Definition 5.5, there is a monotone presentation by operations and equation $\langle \Sigma, \Gamma \rangle$ of the underlying ordinary functor L'_o if $\mathcal{A}_o \rightarrow \mathbf{Pos}_o \rightarrow \mathbf{Set}$ is a variety. This is due to the following result.

Theorem 5.15. *Let L be an endofunctor on a category \mathcal{A} that is both an ordered and an ordinary variety. If L has an ordered presentation, then it has a presentation by monotone operations and equations.*

Proof. To say that \mathcal{A} is both an ordered and an ordinary variety is to say that \mathcal{A} comes equipped with a forgetful functor $\mathcal{A} \rightarrow \mathbf{Pos}$ so that $\mathcal{A} \rightarrow \mathbf{Pos}$ is an ordered variety and $\mathcal{A}_o \rightarrow \mathbf{Pos}_o \rightarrow \mathbf{Set}$ is an ordinary variety. If L has an ordered presentation then it preserves enriched sifted colimits, hence L_o preserves ordinary sifted colimits, hence L_o has a presentation. \square

We can now conclude what we shall need to know about functors on DL.

Theorem 5.16. *For an ordinary functor $L' : \mathbf{DL} \rightarrow \mathbf{DL}$ the following are equivalent.*

- (1) L' has a presentation by monotone operations and equations.
- (2) L' preserves enriched sifted colimits.
- (3) L' is the \mathbf{Pos} -enriched left Kan extension of its restriction to finitely generated free distributive lattices.

As in Proposition 2.4, we now obtain that

Corollary 5.17. *If T' is the the convex powerset functor, then the functor L' of Example 2.6 is isomorphic to the sifted colimits preserving functor \mathbf{L}' whose restriction to \mathbf{DL}_{ff} is $P'T'^{\text{op}}S'$ as in (8).*

6. POSITIVE COALGEBRAIC LOGIC

6.A. Morphisms of logical connections. We recall the (enriched) logical connections (dual adjunctions, see [31]) between sets and Boolean algebras, and between posets and distributive lattices. Both are as **Pos**-enriched, where for the first logical connection the enrichment is discrete. They are related as follows:

$$(26) \quad \begin{array}{ccc} \text{Set}^{\text{op}} & \begin{array}{c} \xleftarrow{S} \\ \perp \\ \xrightarrow{P} \end{array} & \text{BA} \\ D^{\text{op}} \downarrow & & \downarrow W \\ \text{Pos}^{\text{op}} & \begin{array}{c} \xleftarrow{S'} \\ \perp \\ \xrightarrow{P'} \end{array} & \text{DL} \end{array}$$

In the top row of the above diagram, recall again that P is the contravariant powerset functor, while S maps a Boolean algebra to its set of ultrafilters. The bottom row has P' mapping a poset to the distributive lattice of its upper-sets, and S' associating to each distributive lattice the poset of its prime filters. About the pair of functors connecting the two logical connections: D was introduced earlier as the discrete functor, while W is the functor associating to each Boolean algebra its underlying distributive lattice.

It is easy to see that the pair (D^{op}, W) is a *morphism of adjunctions* in the sense of [34, IV.7]. This means that the following diagrams commute, and that the coherence condition below holds:

$$(27) \quad \begin{array}{ccc} \text{Set}^{\text{op}} & \xrightarrow{P} & \text{BA} \\ D^{\text{op}} \downarrow & & \downarrow W \\ \text{Pos}^{\text{op}} & \xrightarrow{P'} & \text{DL} \end{array} \quad \begin{array}{ccc} \text{BA} & \xrightarrow{S} & \text{Set}^{\text{op}} \\ W \downarrow & & \downarrow D^{\text{op}} \\ \text{DL} & \xrightarrow{S'} & \text{Pos}^{\text{op}} \end{array} \quad \epsilon' D^{\text{op}} = D^{\text{op}} \epsilon$$

where ϵ and ϵ' are the counits of $S \dashv P$ and $S' \dashv P'$, respectively.

6.B. Positive coalgebraic logic. We shall now expand the propositional logics **BA** and **DL** by modal operators. We start with a **Set**-endofunctor T in the top left-hand corner of (26). We are mostly interested in the case where $T' : \text{Pos} \rightarrow \text{Pos}$ is the posetification of T (Definition 4.1) and $L : \text{BA} \rightarrow \text{BA}$ and $L' : \text{DL} \rightarrow \text{DL}$ are (the functors of) the associated logics as in (6) and (9), in which case we denote the logics by boldface letters **L** and **L'**.

But some of the following holds under the weaker assumptions that T' is an arbitrary extension of T and that L and L' are arbitrary coalgebraic logics for T and T' , respectively. We therefore let T be a **Set**-endofunctor and T' be an extension of T to **Pos** as in (14). Logics for T, T' are given by functors $L : \text{BA} \rightarrow \text{BA}$ and $L' : \text{DL} \rightarrow \text{DL}$ and natural transformations

$$\delta : LP \rightarrow PT^{\text{op}} \quad \delta' : L'P' \rightarrow P'T'^{\text{op}}$$

Intuitively, δ and δ' assign to the syntax given by (presentations of) L and L' the corresponding one-step semantics in subsets or upper sets. To compare L and L' we need the isomorphism $\alpha : DT \rightarrow T'D$ saying that T' extends T , and also the relation $WP = P'D^{\text{op}}$ from (27) (which formalizes the trivial observation that taking all upsets of a discrete set is the same as taking all subsets). Referring back to the introduction, we now make the following

Definition 6.1. We say that a logic (L', δ') for T' is a *positive fragment* of the logic (L, δ) for T , if there is a natural transformation $\beta : L'W \rightarrow WL$ with $W\delta \circ \beta P = P'\alpha^{\text{op}} \circ \delta'D^{\text{op}}$, or, in diagrams

$$(28) \quad \begin{array}{ccccc} \text{Set}^{\text{op}} & \xrightarrow{P} & \text{BA} & \xrightarrow{W} & \text{DL} \\ T^{\text{op}} \downarrow & \swarrow \delta & \downarrow L & \swarrow \beta & \downarrow L' \\ \text{Set}^{\text{op}} & \xrightarrow{P} & \text{BA} & \xrightarrow{W} & \text{DL} \end{array} = \begin{array}{ccccc} \text{Set}^{\text{op}} & \xrightarrow{D^{\text{op}}} & \text{Pos}^{\text{op}} & \xrightarrow{P'} & \text{DL} \\ T^{\text{op}} \downarrow & \swarrow \alpha^{\text{op}} & \downarrow T'^{\text{op}} \swarrow \delta' & & \downarrow L' \\ \text{Set}^{\text{op}} & \xrightarrow{D^{\text{op}}} & \text{Pos}^{\text{op}} & \xrightarrow{P'} & \text{DL} \end{array}$$

We call (L', δ') the *positive fragment* of (L, δ) if β is an isomorphism.

Recall that we defined the logics \mathbf{L}, \mathbf{L}' induced by T and an extension T' as $\mathbf{L} = PTS$ and $\mathbf{L}' = P'T'^{\text{op}}S'$ on discretely finitely generated free objects. As explained in the introduction, our desired result is to prove that a certain canonically given $\beta : \mathbf{L}'W \rightarrow W\mathbf{L}$, denoted by β , is an isomorphism. The difficulty, as well as the need for the proviso that T preserves weak pullbacks, stems from the fact that in DL (as opposed to BA) the class of functors determined on finitely generated free algebras is strictly smaller than the class of functors determined on finitely presentable (=finite) algebras. As stepping stones, therefore, we first investigate what happens in the cases where the functors L, L' are determined on all algebras and on finitely presentable algebras, before we turn to the situation of functors determined on strongly finitely presentable (=finitely generated free) algebras.

6.C. The case of $L' = P'T'^{\text{op}}S'$ on all algebras. We shall associate to any extension $\alpha : DT \rightarrow T'D$ the pairs (L, δ) and (L', δ') corresponding to T and T' respectively, with $L = PT^{\text{op}}S$ and $\delta = PT^{\text{op}}\epsilon : PT^{\text{op}}SP \rightarrow PT^{\text{op}}$, $L' = P'T'^{\text{op}}S'$ and δ' being defined analogously. We then immediately obtain an isomorphism β by the following:

Proposition 6.2. *Given an extension $\alpha : DT \rightarrow T'D$, the natural isomorphism $\beta : L'W \rightarrow WL$ given by the composite below*

$$\begin{array}{ccccccc} & & L & & & & \\ & \swarrow & & \searrow & & \swarrow & \\ \text{BA} & \xrightarrow{S} & \text{Set}^{\text{op}} & \xrightarrow{T^{\text{op}}} & \text{Set}^{\text{op}} & \xrightarrow{P} & \text{BA} \\ \downarrow W & & \downarrow D^{\text{op}} & \nearrow \alpha^{\text{op}} & \downarrow D^{\text{op}} & & \downarrow W \\ \text{DL} & \xrightarrow{S'} & \text{Pos}^{\text{op}} & \xrightarrow{T'^{\text{op}}} & \text{Pos}^{\text{op}} & \xrightarrow{P'} & \text{DL} \\ & \swarrow & & \searrow & & \swarrow & \\ & & L' & & & \searrow & \end{array}$$

exhibits $L' = P'T'^{\text{op}}S'$ as the positive fragment of $L = PT^{\text{op}}S$.

Proof. This follows from (D^{op}, W) being a morphism of adjunctions (see (27)). \square

6.D. The case of $\bar{L}' = P'T'^{\text{op}}S'$ on finitely presentable algebras. A similar result holds if we define logics via $PT^{\text{op}}SA$ for finitely presentable A , as we are going to show now. To this end, we use the subscript $(-)_f$ to denote the restriction of both categories and (domain-codomain) functors to finite¹⁰ objects as e.g. when

¹⁰As Pos is locally finitely presentable as closed category, and ordinary categories $\text{Set}_o, \text{DL}_o, \text{BA}_o$ are also locally finitely presentable, it follows that the finitely presentable objects in all the above categories are precisely the same as in the ordinary case, i.e. the ones for which the underlying sets are finite [33].

writing the dense inclusions $I : \mathbf{Set}_f \rightarrow \mathbf{Set}$, $I' : \mathbf{Pos}_f \rightarrow \mathbf{Pos}$, $J : \mathbf{BA}_f \rightarrow \mathbf{BA}$ and $J' : \mathbf{DL}_f \rightarrow \mathbf{DL}$. Note that we have the following commuting diagram

$$(29) \quad \begin{array}{ccc} S_f \dashv P_f & \xrightarrow{(D_f^{\text{op}}, W_f)} & S'_f \dashv P'_f \\ (I^{\text{op}}, J) \downarrow & & \downarrow (I'^{\text{op}}, J') \\ S \dashv P & \xrightarrow{(D^{\text{op}}, W)} & S' \dashv P' \end{array}$$

in the category of transformations of adjoints.

Define $(\bar{L}, \bar{\delta})$ for T as $\bar{L} = \text{Lan}_J(PT^{\text{op}}SJ)$ and $\bar{\delta} = \bar{L}P \rightarrow PT^{\text{op}}$ as the adjoint transpose of $\bar{L} \rightarrow PT^{\text{op}}S$ arising from the universal property of the left Kan extension \bar{L} . By construction, \bar{L} is finitary and is given by $PT^{\text{op}}S$ on finite(ly presentable) Boolean algebras. Similarly, obtain $(\bar{L}', \bar{\delta}')$ for T' on distributive lattices.

Since W is left adjoint (the enriched right adjoint of W sending a distributive lattice A to the Boolean algebra of complemented elements in A , also known as the center of A), in particular it is finitary. Thus to obtain an (iso)morphism $\bar{\beta} : \bar{L}'W \rightarrow W\bar{L}$ between finitary functors, it suffices to have its restriction along J to finitely presentable objects. But we can get such a transformation from the isomorphism of Prop. 6.2, namely

$$(30) \quad \beta_f : \bar{L}'WJ \cong \bar{L}'J'W_f \cong P'T'^{\text{op}}S'J'W_f \cong WPT^{\text{op}}SJ \cong W\bar{L}J$$

where the second and the last isomorphisms are provided by the units of left Kan extensions.

Recall the definition of \mathbf{L} from (6). Since every finitely presentable non-trivial Boolean algebra is a retract of a finitely generated free algebra, we can identify $\mathbf{L} = \bar{L}$ (see e.g. [29, Proposition 3.4]). To summarize, we have

Proposition 6.3. *The isomorphism $\bar{\beta}$ exhibits $\bar{L}' = P'T'^{\text{op}}S'J'$ as the maximal positive fragment of (\mathbf{L}, δ) .*

Proof. The easiest way to check (28) is to show that $\bar{\beta}_f$, defined by (30), fulfills

$$(31) \quad \begin{array}{ccccc} & & \mathbf{Pos}_f^{\text{op}} & & \\ & D_f^{\text{op}} \nearrow & & \nwarrow P'_f & \\ \mathbf{Set}_f^{\text{op}} & & & & \mathbf{DL}_f \\ & P_f \searrow & & \nearrow W_f & \\ & & \mathbf{BA}_f & & \\ T^{\text{op}}I^{\text{op}} \downarrow & & \downarrow \delta_f I^{\text{op}} & & \downarrow \beta_f \\ \mathbf{Set}^{\text{op}} & & & & \mathbf{DL} \\ & P \searrow & & \nearrow W & \\ & & \mathbf{BA} & & \end{array} \quad = \quad \begin{array}{ccccc} & & \mathbf{Pos}_f^{\text{op}} & & \\ & D_f^{\text{op}} \nearrow & & \nwarrow P'_f & \\ \mathbf{Set}_f^{\text{op}} & & & & \mathbf{DL}_f \\ & \downarrow \alpha^{\text{op}} I^{\text{op}} & & \downarrow \delta'_f I'^{\text{op}} & \\ & & \mathbf{Pos}^{\text{op}} & & \\ T^{\text{op}}I^{\text{op}} \downarrow & & D^{\text{op}} \nearrow & & \nwarrow P' \\ \mathbf{Set}^{\text{op}} & & & & \mathbf{DL} \\ & P \searrow & & \nearrow W & \\ & & \mathbf{BA} & & \end{array}$$

But this follows from Prop. 6.2 and (29). \square

This proposition does not yet give us the desired result, as \bar{L}' is not necessarily determined by its action on discretely finitely generated free algebras and, therefore, need not give rise to a variety of modal algebras. The next paragraph will investigate when \bar{L}' does actually have this additional property.

6.E. The case of $\mathbf{L}' = P'T'^{\text{op}}S'$ on discretely finitely generated free algebras. Recall that we denoted by $\mathbf{J} : \mathbf{BA}_{\text{ff}} \rightarrow \mathbf{BA}$ and $\mathbf{J}' : \mathbf{DL}_{\text{ff}} \rightarrow \mathbf{DL}$ the inclusion functors of the full subcategories spanned by the algebras which are free on finite discrete posets.

Definition 6.4. Let T' be a \mathbf{Pos} -endofunctor. We define the logic for T' to be the pair (\mathbf{L}', δ') , where:

- $\mathbf{L}' : \mathbf{DL} \rightarrow \mathbf{DL}$ is a \mathbf{Pos} -functor preserving sifted colimits, whose restriction to free discretely finitely generated distributive lattices is $P'T'^{\text{op}}S'\mathbf{J}'$, that is, $\mathbf{L}' = \text{Lan}_{\mathbf{J}'}(P'T'^{\text{op}}S'\mathbf{J}')$.

$$\begin{array}{ccccc} \mathbf{DL}_{\text{ff}} & \xrightarrow{\quad \mathbf{J}' \quad} & \mathbf{DL} & & \\ \downarrow \mathbf{J}' & \nearrow & \downarrow \mathbf{L}' & & \\ \mathbf{DL} & \xrightarrow{P'} \mathbf{Pos}^{\text{op}} \xrightarrow{T'^{\text{op}}} \mathbf{Pos}^{\text{op}} \xrightarrow{S'} & \mathbf{DL} & & \end{array}$$

- $\delta' : \mathbf{L}'P' \rightarrow P'T'^{\text{op}}$ is the pasting composite

$$(32) \quad \begin{array}{ccccc} \mathbf{DL} & \xrightarrow{S'} \mathbf{Pos}^{\text{op}} \xrightarrow{T'^{\text{op}}} \mathbf{Pos}^{\text{op}} & & & \\ \downarrow \mathbf{L}' & \nearrow & \downarrow P' & \nearrow \varepsilon' & \\ \mathbf{DL} & \xrightarrow{\quad \quad} \mathbf{DL} & \xrightarrow{S'} \mathbf{Pos}^{\text{op}} & & \end{array}$$

that is, the adjoint transpose of $\mathbf{L}' \rightarrow P'T'^{\text{op}}S'$ given by the universal property of the left Kan extension \mathbf{L}' .

Remark 6.5. By the above definition, \mathbf{L}' preserves sifted colimits. Thus, by Theorem 5.8, \mathbf{L}' has an equational presentation by monotone operations, which in turn gives rise to a positive modal logic concretely given in terms of modal operators and axioms.

Recall that $\bar{\mathbf{L}}' = P'T'^{\text{op}}S'$ on finitely presentable (=finite) distributive lattices and that $\mathbf{L}' = P'T'^{\text{op}}S'$ on discretely finitely generated free algebras.

Theorem 6.6. *Let T be a \mathbf{Set} -endofunctor and T' a \mathbf{Pos} -extension of T which preserves coreflexive inserters. Then $(\bar{\mathbf{L}}', \bar{\delta}')$ and (\mathbf{L}', δ') coincide. In particular, it follows from Proposition 6.3 that \mathbf{L}' is the positive fragment of \mathbf{L} .*

Remark 6.7. The isomorphism $(\bar{\mathbf{L}}, \bar{\delta}) \cong (\mathbf{L}, \delta)$ of the corresponding Boolean logic for \mathbf{Set} -functors was established in [29]. (Recall that \mathbf{L} was introduced in (6), while $\bar{\mathbf{L}}$ appeared in Paragraph 6.D above.)

Proof of Thm. 6.6. In order to show $(\bar{\mathbf{L}}', \bar{\delta}') \cong (\mathbf{L}', \delta')$, it is enough to check that $\bar{\mathbf{L}}'$ and \mathbf{L}' agree on finite distributive lattices, as both are finitary. That is, we need to show that \mathbf{L}' is $P'T'^{\text{op}}S'$ on any finite distributive lattice, not just on the free lattices with finitely many discrete generators. In particular, this will also imply $\bar{\delta}' \cong \delta'$.

(1) Using that the free-distributive lattice monad $U'F' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ is strongly finitary, one can exhibit every (finite) distributive lattice as a coinserter of free (finite) ones (because an equation can be expressed by (pairs of) inequations).

Namely, take A to be a finite distributive lattice and consider the counit $\varepsilon_A : F'U'A \rightarrow A$ in \mathbf{DL} . It is surjective, called an **so-morphism** in [32], hence a coinserter of some pair $A' \rightrightarrows F'U'A$ (by factoring the pair through its image, we can assume without loss of generality that A' is finite). Now post-compose this pair with $\varepsilon_{A'} : F'U'A' \rightarrow A'$ to obtain $F'U'A' \rightrightarrows F'U'A$.

Again, since $\varepsilon_{A'}$ is surjective, it is a coinserter. Hence $F'U'A' \rightrightarrows F'U'A$ and $A' \rightrightarrows F'U'A$ have the same “coinserter cocones”. This exhibits A as the coinserter of $F'U'A' \rightrightarrows F'U'A$.

(2) We need to check that \mathbf{L}' and $\bar{\mathbf{L}}'$ agree on all free distributive lattices on finite posets.

Given a finite poset X , exhibit it as a reflexive coinserter as in (12):

$$(33) \quad \begin{array}{c} \text{Di} \\ \curvearrowright \\ DX_1 \xrightleftharpoons[Dd_1]{Dd_0} DX_0 \xrightarrow{c} X \end{array}$$

Apply now $\mathbf{L}'F'$ to (33); we get

$$(34) \quad \begin{array}{ccc} \begin{array}{c} \text{L}'F'Di \\ \curvearrowright \\ \text{L}'F'DX_1 \xrightleftharpoons[\text{L}'F'Dd_1]{\text{L}'F'Dd_0} \text{L}'F'DX_0 \end{array} & \xrightarrow{\text{L}'F'q} & \text{L}'F'X \\ \downarrow \cong & & \downarrow \cong \\ \begin{array}{c} P'T'^{\text{op}}[DX_1, 2] \xrightleftharpoons[P'T'^{\text{op}}[Dd_1, 2]]{P'T'^{\text{op}}[Dd_0, 2]} P'T'^{\text{op}}[DX_0, 2] \end{array} & & P'T'^{\text{op}}[X, 2] \\ \uparrow \text{P'T'^{op}[Di, 2]} & & \end{array}$$

The upper row of the diagram is again a coinserter, as F' is a left adjoint and \mathbf{L}' preserves sifted colimits by definition. Remember that both \mathbf{L}' and $\bar{\mathbf{L}}'$ are isomorphic to $P'T'^{\text{op}}S'$ on free lattices with finitely many discrete generators, and notice that $S'F' \cong [-, 2]$. Consequently, we just need to show that the coinserter is (isomorphic to) $P'T'^{\text{op}}S'F'X \cong P'T'^{\text{op}}[X, 2]$.

In order to achieve this, use first that S' and F' are left adjoints and move the diagram (33) and its colimit from \mathbf{DL} to \mathbf{Pos}^{op} . Thus in \mathbf{Pos} , we obtain $[c, 2]$ as the (coreflexive) inserter of the pair $[Dd_0, 2], [Dd_1, 2]$. By hypothesis, T' preserves it. Now the desired result follows from Lemma 6.8. \square

Lemma 6.8. *The functor $P' : \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{DL}$ preserves reflexive inserters.*

In order to prove this rather technical result, we need some preliminaries.

Lemma 6.9. *Let $e : E \rightarrow X$ be an embedding (i.e. a monotone and order reflecting map) of posets. Then $[e, 2]$ has a right inverse.*

Proof. Remember that any poset can be seen as a category enriched over the two-elements poset 2 , and any monotone map $e : E \rightarrow X$ as a 2 -enriched functor. Pre-composition with e gives a functor between posets $[e, 2] : [X, 2] \rightarrow [E, 2]$ which always has a left (and a right) adjoint, given by left (and right) Kan extensions. Explicitly, the left adjoint $\exists_e : [E, 2] \rightarrow [X, 2]$ maps an upper set to the up-set closure of its image via e .

Notice that e is an embedding precisely when it is fully faithful as a $\mathbb{2}$ -enriched functor, thus by [23], Proposition 4.23 the unit of the adjunction (the natural transformation corresponding to left Kan extensions) is an isomorphism. But in $\mathbb{2}$ -enriched categories (posets), this means equality, thus $[e, \mathbb{2}] \circ \exists_e = \text{id}_{[E, \mathbb{2}]}$. \square

Remember that we introduced in Definition 4.10 the notion of exact square. We shall give now an equivalent formulation:

Lemma 6.10 (Beck-Chevalley property). *The diagram (17) exhibits an exact square iff $[g, \mathbb{2}] \circ \exists_f = \exists_\alpha \circ [\beta, \mathbb{2}]$:*

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & X \\ \beta \downarrow & \swarrow & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} [E, \mathbb{2}] & \xleftarrow{[\alpha, \mathbb{2}]} & [A, \mathbb{2}] \\ \exists_\beta \downarrow & = & \downarrow \exists_f \\ [Y, \mathbb{2}] & \xleftarrow{[g, \mathbb{2}]} & [Z, \mathbb{2}] \end{array}$$

Proof. It follows easily by direct computation. \square

Now we have all ingredients to prove Lemma 6.8:

Proof of Lemma 6.8. Notice that $U'P' = [-, \mathbb{2}]$ and that U' is strongly finitary monadic. Since DL is an ordered variety, it has all (sifted) colimits, in particular reflexive coinserter, and U' creates them. Thus it is enough to show that $[-, \mathbb{2}] : \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{Pos}$ preserves reflexive coinserter.

But reflexive coinserter in \mathbf{Pos}^{op} are (coreflexive) inserters; consider therefore two monotone maps with common left inverse in \mathbf{Pos}

$$(35) \quad \begin{array}{ccc} & i & \\ \swarrow & & \searrow \\ X & \xrightleftharpoons[g]{f} & Y \end{array}$$

The inserter of the above data is realized as $\text{ins}(f, g) = \{x \in X \mid f(x) \leq g(x)\}$, together with the inclusion map $\text{ins}(f, g) \xrightarrow{e} X$. In particular, the diagram below is an exact square:

$$\begin{array}{ccc} \text{ins}(f, g) & \xrightarrow{e} & X \\ e \downarrow & \swarrow & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

By Lemma 6.10, we obtain $[g, \mathbb{2}] \circ \exists_f = \exists_e \circ [e, \mathbb{2}]$. As both e and f are embeddings (f being a split mono), $[e, \mathbb{2}] \circ \exists_e = \text{id}_{[\text{ins}(f, g), \mathbb{2}]}$ and $[f, \mathbb{2}] \circ \exists_f = \text{id}_{[X, \mathbb{2}]}$ by Lemma 6.9.

That is, by applying $[-, \mathbb{2}]$ to the diagram (35) augmented by $\text{ins}(f, g) \xrightarrow{e} X$ and \exists_e, \exists_f , produces $[\text{ins}(f, g), \mathbb{2}]$ as the split (thus absolute) coinserter of $[f, \mathbb{2}], [g, \mathbb{2}]$.

$$\begin{array}{ccccc} & [i, \mathbb{2}], \exists_f & & & \\ \swarrow & & \searrow & & \\ [Y, \mathbb{2}] & \xrightleftharpoons[g, \mathbb{2}]{[f, \mathbb{2}]} & [X, \mathbb{2}] & \xrightarrow{[e, \mathbb{2}]} & [\text{ins}(f, g), \mathbb{2}] \\ & & \nwarrow & \searrow & \\ & & & \exists_e & \end{array}$$

Therefore $[-, \mathbb{2}]$ maps coreflexive inserters to (reflexive and split) coinserter. \square

The next lemma shows that for a locally monotone functor on \mathbf{Pos} , preservation of exact squares entails the condition needed in Theorem 6.6, namely the preservation of coreflexive inserters:

Lemma 6.11. *If T' is a locally monotone functor on \mathbf{Pos} which preserves exact squares, then it preserves embeddings and coreflexive inserters.*

Proof. The first assertion follows from the observation [19] that each embedding $e : X \rightarrow Y$ can be realized as an exact square, namely

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ \text{id} \downarrow & \swarrow & \downarrow e \\ X & \xrightarrow{e} & Y \end{array}$$

For the second one, consider

$$\text{ins}(f, g) \xrightarrow{e} X \begin{array}{c} \xrightarrow{i} \\ \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

a (coreflexive) inserter. In particular,

$$\begin{array}{ccc} \text{ins}(f, g) & \xrightarrow{e} & X \\ e \downarrow & \swarrow & \downarrow f \\ X & \xrightarrow{g} & Y \end{array}$$

is an exact square as remarked in the proof of Lemma 6.8, thus T' maps it to the exact square

$$\begin{array}{ccc} T'\text{ins}(f, g) & \xrightarrow{T'e} & T'X \\ T'e \downarrow & \swarrow & \downarrow T'f \\ T'X & \xrightarrow{T'g} & T'Y \end{array}$$

Let now $u : U \rightarrow T'X$ a monotone map such that $T'f \circ u \leq T'g \circ u$. For each $x \in U$, $T'f(u(x)) \leq T'g(u(x))$, thus there is some $w \in T'\text{ins}(f, g)$ with $u(x) \leq T'e(w)$ and $T'e(w) \leq u(x)$, that is, $u(x) = T'e(w)$. As $T'e$ is an embedding, such element w is uniquely determined. Moreover, the assignment $x \mapsto w$ is monotone, as if $x_1 \leq x_2$, then $T'e(w_1) = u(x_1) \leq u(x_2) = T'e(w_2)$ and $T'e$ is again an embedding as shown earlier, hence $w_1 \leq w_2$. This covers the 1-dimensional aspect of inserters. For the remaining, use one more time that $T'e$ is an embedding. \square

As a consequence of all the results of this section and of Proposition 4.11, we obtain

Theorem 6.12. *Let $T : \mathbf{Set} \rightarrow \mathbf{Set}$ be a weak-pullback preserving functor and $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ its posetification. Let (\mathbf{L}, δ) and (\mathbf{L}', δ') be the associated logics of T and T' , that is $\mathbf{L} = \text{Lan}_{\mathbf{J}}(PT^{\text{op}}S\mathbf{J})$ and $\mathbf{L}' = \text{Lan}_{\mathbf{J}'}(P'T'^{\text{op}}S'\mathbf{J}')$. Then (\mathbf{L}', δ') is the positive fragment of (\mathbf{L}, δ) .*

Our introductory example of positive modal logic is now regained as an instance of this theorem. It can also easily be adapted to Kripke polynomial functors. More interesting are the cases of probability distribution functor and of multiset functor. We know from the theorem above that they have maximal positive fragments, but their explicit descriptions still needs to be worked out.

Let us conclude with an example showing what goes wrong for an extension that does not preserve weak pullbacks.

Example 6.13. For $T = \text{Id}$, the corresponding finitary logics is $\mathbf{L} = \text{Id}$ on \mathbf{BA} , with trivial semantics $\delta : LP \rightarrow PT^{\text{op}}$. It was noticed in Remark 4.2(1) that the identity functor also admits as extension the discrete connected components functor $T' = DC$. But the latter does not preserve embeddings, nor coreflexive inserters. The corresponding logic \mathbf{L}' for T' is given by the constant functor to the distributive lattice $\mathbf{2}$. Thus $\beta : \mathbf{L}'W \rightarrow W\mathbf{L}$ fails to be an isomorphism (it is just the unique morphism from the initial object).

7. MONOTONE PREDICATE LIFTINGS

In this section we show that the logic of the posetification T' of T coincides with the logic of all monotone predicate liftings of T .

Recall from [36, 38] that a predicate lifting of arity n for T is a natural transformation

$$\heartsuit : \mathbf{Set}(-, 2^n) \rightarrow \mathbf{Set}(T-, 2)$$

Using the adjunction $D \dashv V : \mathbf{Pos} \rightarrow \mathbf{Set}$, a predicate lifting can be described as a natural transformation

$$\heartsuit' : \mathbf{Pos}(D-, [Dn, \mathbf{2}]) \rightarrow \mathbf{Pos}(DT-, \mathbf{2})$$

It is called *monotone* if each component is monotone (as a map between hom-posets). By Yoneda lemma, one can also identify a predicate lifting with an map $\heartsuit : T(2^n) \rightarrow \mathbf{2}$. Then the above simply says that \heartsuit is monotone if for all $\overline{a_1} \leq \overline{a_2} : DX \rightarrow [Dn, \mathbf{2}]$, we have that $\heartsuit \circ T\overline{a_1} \leq \heartsuit \circ T\overline{a_2}$, where $\overline{f} : DX \rightarrow Y$ denotes the adjoint transpose of $f : X \rightarrow VY$.

Consider now a locally monotone \mathbf{Pos} -functor T' and a finite poset p . By mimicking the above, we define a predicate lifting for T' of arity p as being a \mathbf{Pos} -natural transformation

$$\heartsuit' : \mathbf{Pos}(-, [p, \mathbf{2}]) \rightarrow \mathbf{Pos}(T'-, \mathbf{2})$$

which again can be identified with $\heartsuit' \in \mathbf{Pos}(T'([p, \mathbf{2}]), \mathbf{2})$.

Proposition 7.1. *Let T be a \mathbf{Set} -functor and $T' : \mathbf{Pos} \rightarrow \mathbf{Pos}$ its posetification. Then there is a bijection between the predicate liftings of T' of discrete arity Dn and the monotone predicate liftings of T of arity n , for each finite n .*

Proof. Let p be an arbitrary finite poset. Consider the composition of the two following monomorphisms:

$$(36) \quad \mathbf{Pos}(T'([p, \mathbf{2}]), \mathbf{2}) \rightarrow \mathbf{Set}(VT'([p, \mathbf{2}]), V\mathbf{2}) \rightarrow \mathbf{Set}(TV([p, \mathbf{2}]), V\mathbf{2})$$

The first arrow above is monic by faithfulness of V . The second one is also, as it is given by pre-composition with the natural epimorphism $\tau : TV \rightarrow VT'$ (the mate of the isomorphism $\alpha : DT \rightarrow T'D$ under the adjunction $D \dashv V$). The latter is indeed epic because for each poset X , τ_X is exactly the coinserter map $TX_0 \rightarrow T'X$.

In case the arity is discrete as $p = Dn$, notice that by $V([Dn, 2]) = 2^n$, the right hand side of Equation (36) is precisely $\text{Set}(T(2^n), 2)$. A predicate lifting $\heartsuit' \in \text{Pos}(T'([Dn, 2]), 2)$ is then sent to $\lambda := V\heartsuit' \circ \tau_{[Dn, 2]} : T(2^n) \rightarrow 2$. Let $a : X \rightarrow 2^n = V([Dn, 2])$. Then $\overline{\lambda \circ Ta} = \heartsuit' \circ T'(\overline{a}) \circ \alpha_X$ (by chasing diagrams) and the monotonicity of λ follows now easily. Thus the predicate liftings of T' of discrete arity are among the monotone predicate liftings for T .

To show the inverse correspondence, recall one more time that the posetification T' is constructed as a coinsserter (Thm. 4.3). Let $\heartsuit : T(2^n) \rightarrow 2$ be a predicate lifting for T . Then, from the universal property of coinserters, one can easily check that $\overline{\heartsuit} : DT(2^n) \rightarrow 2$ factorize to a predicate lifting for T' of discrete arity, $T'([Dn, 2]) \rightarrow 2$, iff \heartsuit is monotone in the sense mentioned above. In more detail: let X_0 be the set $2^n = V[Dn, 2]$; that is, the underlying set of the poset $[Dn, 2]$, and X_1 the underlying set of the order on $[Dn, 2]$, with projections denoted as usual $d_0, d_1 : X_1 \rightarrow X_0$. Then with notations as above, one has $\overline{d_0} \leq \overline{d_1}$; thus if \heartsuit is monotone, this entails $\overline{\heartsuit} \circ Td_0 \leq \overline{\heartsuit} \circ Td_1$, thus $\overline{\heartsuit} : DTX_0 \rightarrow 2$ factorizes in Pos to a predicate lifting for T' of discrete arity $\heartsuit' : T'[Dn, 2] \rightarrow 2$.

$$\begin{array}{ccccc}
 T'DX_1 & \xrightleftharpoons[T'Dd_1]{T'Dd_0} & T'DX_0 & & \\
 \downarrow \alpha_{X_1}^{-1} & & \downarrow \alpha_{X_0}^{-1} & \searrow T'\epsilon_{[Dn, 2]} & \\
 DTX_1 & \xrightleftharpoons[DTd_1]{DTd_0} & DTX_0 & \xrightarrow{e} & T'[Dn, 2] \\
 & & \downarrow \overline{\heartsuit} & \swarrow \heartsuit' & \\
 & & 2 & &
 \end{array}$$

From the above diagram we have that $\heartsuit' \circ T'\epsilon_{[Dn, 2]} \circ \alpha_{V[Dn, 2]} = \overline{\heartsuit}$, thus we see we can recover the original monotone predicate lifting for T :

$$\lambda = V\heartsuit' \circ \tau_{[Dn, 2]} = V\heartsuit' \circ VT'\epsilon_{[Dn, 2]} \circ V\alpha_{V[Dn, 2]} = V\overline{\heartsuit} = \heartsuit$$

Finally, we note that we used the assumption that T' is the posetification of T in order to have an extension such that $TV \rightarrow VT'$ is epi. \square

As a corollary, we obtain

Theorem 7.2. *Let T be a Set-functor. If the posetification T' of T preserves embeddings, then the logic of all monotone predicate liftings of T is expressive.*

Remark 7.3. We know from Prop. 4.11 that if T preserves weak pullbacks then T' preserves exact squares, thus also embeddings. So the above applies to weak-pullbacks preserving functors. This result was obtained in [28, Corollary 6.9] already in a different way for finitary functors.

8. CONCLUSIONS

In the area of semantics of programming languages one encounters a wide variety of base categories including various metric spaces and complete partial orders. It would be of interest to draw the landscape of these different categories together with a toolkit connecting them. This paper can be seen as a rudimentary effort in

this direction. Indeed, we relate systems and their logics across the morphism of connections

$$(S \dashv P : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{BA}) \longrightarrow (S' \dashv P' : \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{DL}).$$

Moreover, we transfer functors along this morphism via left Kan-extensions and characterize the functors that arise in that way as those preserving certain classes of colimits. Finally, we have shown how results about modal logics can be derived from such a framework. It will be interesting to explore whether similar techniques apply to more sophisticated domains than \mathbf{Set} and \mathbf{Pos} .

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